Poker Hand Probabilities

The inventors of the card game poker may or may not have had mathematical training, but the evidence suggests they might have been able to pass a probability class. For one thing, the different hands possible in poker are correctly ranked from the best hand (a royal flush), the least probable hand, down to the worst hand (high card or “garbage”), the most probable. We will do some example calculations of the probabilities in 5-card poker (for example, “5-card stud”). But first, notice that in every case the pool of possible hands (the sample space) has exactly \( \binom{52}{5} = \frac{52!}{5!47!} = 2,598,960 \) hands. To understand this handout you must have some knowledge of counting rules, including \( \binom{n}{k} \) and the multiplication rule (sometimes called the Fundamental Principle of Counting).

Royal Flush: A royal flush consists of the top five cards, all in one suit, for example ♦AKQJT (I use T for the 10 of a suit). There are only four different possible royal flushes, one in each suit, so the probability of being dealt one is \( \frac{4}{\binom{52}{5}} \approx 1.54 \times 10^{-6} \). Playing 100 hands of poker per day you would expect this hand about once every 18 years.

Straight Flush: A straight flush consists of five cards, all of the same suit, and in rank order (so it’s a straight and a flush). Examples: ♥7,8,9,T,J or ♦4,5,6,7,8. To count these, notice that if I tell you the lowest card of a straight flush (say ♦4) you know the whole hand - it’s the second example hand. The lowest card can be anything below a ten, (it could be an Ace, because A2345 is a straight) so there are ten possible starter cards per suit, altogether 40 starter cards, so altogether 40 straight flushes. Probability is \( \frac{40}{\binom{52}{5}} \approx 1.54 \times 10^{-6} \), exactly 10 times as likely as a royal flush.

Four of a Kind: To have four of a kind, you need all four cards of a certain rank in your hand - for example, all four jacks or all four 6’s. It’s a common mistake to think that since there are 13 ranks there are exactly 13 four-of-a-kind hands; but remember that a poker hand has five cards. A four-of-a-kind hand, like ♦K, ♦K, ♦K, ♦K, ♦8 has a fifth “garbage” card that doesn’t affect the value of the hand: however, each of the 13 foursomes has 48 possible garbage cards to make up the five-card hand. So there are \( 13 \times 48 = 624 \) possible four-of-a-kind hands. Probability: \( \frac{624}{\binom{52}{5}} \approx .00024 \)

Full House: A full house requires three cards of the same rank and a pair of a second rank, for example ♦9, ♦9, ♦9, ♦7, ♦7. To describe a full house precisely you need four pieces of information:

- the rank of the triplet
- which three cards of that rank are used
- the rank of the pair
- which two cards of that rank are used.

We can count the number of ways to answer each of these questions and multiply the answers to get the total number of full houses. There are 13 ranks; there are \( \binom{4}{3} = 4 \) ways to choose 3 cards of a given rank; there are 12 (not 13) ranks left for the pair; and there are \( \binom{4}{2} = 6 \) ways to choose a pair of a given rank. Multiplying all these together there are \( 13 \times 4 \times 12 \times 6 = 3744 \) possible full houses, with a probability of \( \frac{3744}{\binom{52}{5}} \approx .00014 \), about 14 times per 100,000 poker hands.

One Pair: We will skip several types of poker hands to get to perhaps the most difficult calculation, the probability of getting one pair. To get one pair, we need two cards of one rank and three cards that do not match that rank, or each other’s rank, like: ♥Q, ♥Q, ♦8, ♦3, ♦J. To find a pair we need to pick a rank (13 ways) then pick two of the four cards of that rank (\( \binom{4}{2} = 6 \) ways): so there are \( 13 \times 6 = 78 \) two-card pairs. To choose the “garbage” cards, we need to choose 3 different ranks, out of 12 ranks we haven’t chosen yet: this we can do \( \binom{12}{3} = 220 \) ways; and then, for each rank, choose one out of the four cards of that rank, which we can do in \( 4 \times 4 \times 4 = 64 \) ways. Putting it all together, there are \( 78 \times 220 \times 64 \) different hands with one pair, a total of 1,098,240. The probability is thus \( \frac{1098240}{\binom{52}{5}} \approx 0.422 \), so over 40% of hands have one pair.

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