**Detailed Example: Solving a System of Linear Equations by Gaussian Elimination**

To solve a system of equations using Gaussian elimination,

1. Write the equations as an augmented matrix
2. Use the three legal row operations to reduce the matrix to row echelon form (REF) or reduced row echelon form (RREF). The three legal operations are:
   - (a) Multiply any row by a nonzero constant
   - (b) Switch the position of any two rows
   - (c) Add a nonzero multiple of any row to any other row.
3. Interpret the matrix as a simple system of equations and solve.

We’re going to solve the following system of equations in detail.

\[
\begin{align*}
x_1 + 4x_2 + x_3 &= -1 \\
4x_1 + 13x_2 - 5x_3 &= -7 \\
2x_1 + 7x_2 + 2x_3 &= 0
\end{align*}
\]

(1) The augmented matrix (using the coefficients and the right hand side numbers) is:

\[
\begin{pmatrix}
1 & 4 & 1 & -1 \\
4 & 13 & -5 & -7 \\
2 & 7 & 2 & 0
\end{pmatrix}
\]

(2) The strategy is to work from right to left. In each column, get a 1 entry (in a row that has nothing but zeros to the left of this column) and use the 1 entry and rule 2(c) to make every other entry in that column zero; then on to the next column. Our ideal matrix looks like this:

\[
\begin{pmatrix}
1 & 0 & 0 & * \\
0 & 1 & 0 & * \\
0 & 0 & 1 & *
\end{pmatrix}
\]

where the *’s stand for whatever number shows up.

Starting calculations, we see that our augmented matrix already has a 1 in the (1, 1) position (first row, first column) where we want it. We need to clear the rest of the first column by adding multiples of the first row, featuring that 1, to the other rows. First, add \((-4)\) times row 1 to row 2:

\[
\begin{pmatrix}
1 & 4 & 1 & -1 \\
0 & -3 & -9 & -3 \\
2 & 7 & 2 & 0
\end{pmatrix}
\]

Then for the same reason, add \((-2)\) times row 1 to row 3:

\[
\begin{pmatrix}
1 & 4 & 1 & -1 \\
0 & -3 & -9 & -3 \\
0 & -1 & 0 & 2
\end{pmatrix}
\]

Now the first column is perfect. We want to get a 1 in the (2, 2) position: there are at least two ways to do this. We could divide row 2 by \(-3\) (technically, we’re multiplying row 2 by \(-\frac{1}{3}\)), or we could switch rows 2 and 3 then multiply the new row 2 by \(-1\). We’ll make the second choice now, but either way would work fine. Switch:

\[
\begin{pmatrix}
1 & 4 & 1 & -1 \\
0 & -1 & 0 & 2 \\
0 & -3 & -9 & -3
\end{pmatrix}
\]

Now multiply row 2 by \(-1\):

\[
\begin{pmatrix}
1 & 4 & 1 & -1 \\
0 & 1 & 0 & -2 \\
0 & -3 & -9 & -3
\end{pmatrix}
\]
We're still busy with column 2 (remember, work left to right, one column at a time). We need 0's above and below the 1. So we’ll add $-4$ times column 2 to column 1:

$$
\begin{pmatrix}
1 & 0 & 1 & 7 \\
0 & 1 & 0 & -2 \\
0 & -3 & -9 & -3
\end{pmatrix}
$$

Then get a 0 below the 1 by adding 3 times row 2 to row 3:

$$
\begin{pmatrix}
1 & 0 & 1 & 7 \\
0 & 1 & 0 & -2 \\
0 & 0 & -9 & -9
\end{pmatrix}
$$

Now we’re up to column 3: we need a 1 in the (3,3) position, so we’ll multiply row 3 by $-\frac{1}{9}$ (there's really no choice here):

$$
\begin{pmatrix}
1 & 0 & 1 & 7 \\
0 & 1 & 0 & -2 \\
0 & 0 & 1 & 1
\end{pmatrix}
$$

Lastly, we need 0's above our new 1. Row 2 is already fine, but we need to add $-1$ times row 3 to row 1 to clear that nonzero entry.

$$
\begin{pmatrix}
1 & 0 & 0 & 6 \\
0 & 1 & 0 & -2 \\
0 & 0 & 1 & 1
\end{pmatrix}
$$

(3) At this point, we have the matrix in RREF, as desired. We can reinterpret the matrix as a system of equations, as follows:

\[ x_1 = 6 \]
\[ x_2 = -2 \]
\[ x_3 = 1 \]

...which gives us the solution to the system.

What can go wrong?

That equation had a unique solution. With Gaussian elimination we find the unique solution if there is one. If there is no solution, we find that too: one of the equations we come up with will say 0 = 1. The most difficult case is when we have many solutions. For example, a matrix like

$$
\begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 1 & 5 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

leads to a set of equations:

\[ x_1 - x_3 = 0 \]
\[ x_2 + x_3 = 5 \]
\[ 0 = 0 \]

Here, $x_3$ never got to have a leading 1 in a row. This is a symptom of infinitely many solutions: in this case, any choice of a value for the “free” variable $x_3$ leads to a solution. (In general, every variable with no leading 1 can have a value chosen at random; the other variables adjust to make the solution work). If we choose to call the value of $x_3 = t$, we can solve for $x_1$ and $x_2$ in terms of $t$:

\[ x_1 = t \]
\[ x_2 = 5 - t \]
\[ x_3 = t \]

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