Rationalizing/Realizing the Denominator

When a number involves a square root or higher root in the denominator, it is often helpful to rationalize the number, transferring the radical to the numerator. If this is not done, it’s sometimes hard to tell whether two numbers are the same, or different! For example, which two out of these five numbers are the same?

\[
\frac{1}{\sqrt{2} - 1} \quad \sqrt{2} + 1 \quad \sqrt[3]{3} \quad \frac{2}{\sqrt{6}} \quad \frac{\sqrt{6}}{3}
\]

In fact, the first two numbers are equal - and the last three numbers are all equal! This is not obvious at first glance.

Rationalization Techniques

All rationalization techniques involve multiplying by a carefully chosen version of the number 1: we multiply top and bottom by the same expression, which is chosen to cancel out the radical in the denominator.

**Single Square Root in the Denominator** In this case, multiply top and bottom by the same square root in the denominator (sometimes, you can use a smaller square root by ignoring square factors of the denominator: see the second example).

Examples:
- \( \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3} \)
- \( \frac{1}{\sqrt{12}} = \frac{1}{\sqrt{12}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{\sqrt{36}} = \frac{\sqrt{3}}{12} \)

**Single Cube (or higher) Root in the Denominator**

In this case, multiply top and bottom by the smallest cube root that will make the new denominator the cube root of a perfect cube: to do this, factor the number and see what you need to make every prime factor have a power divisible by 3. For example, 72 = 8 * 9 = 2^3 * 3^2; we already have a cube of 2’s, and just need to multiply by 3 to get the perfect cube 216 = 2^3 * 3^3 = 6^3, so

\[
\frac{5}{\sqrt[3]{72}} = \frac{5}{\sqrt[3]{72}} \cdot \frac{\sqrt[3]{3}}{\sqrt[3]{3}} = \frac{5 \sqrt[3]{3}}{\sqrt[3]{216}} = \frac{5 \sqrt[3]{3}}{6}
\]

**Square Root Plus or Minus Something in the Denominator**

In a problem like \( \frac{1}{\sqrt{3}+5} \) we use the algebra fact that \((a+b)(a-b) = a^2 - b^2\). We multiply (top and bottom, as usual) by the *conjugate* expression made from the denominator by replacing + with − or vice versa.

\[
\frac{1}{\sqrt{3} + 5} = \frac{1}{(\sqrt{3} + 5)} \cdot \frac{\sqrt{3} - 5}{\sqrt{3} - 5} = \frac{\sqrt{3} - 5}{(\sqrt{3})^2 - 25} = \frac{\sqrt{3} - 5}{3 - 25} = \frac{5 - \sqrt{3}}{22}
\]

**Cube Root Plus or Minus Something in the Denominator**

Similarly, we use the sum or difference of cubes factoring facts:
- Difference of cubes: \( a^3 - b^3 = (a-b)(a^2 + ab + b^2) \)
- Sum of cubes: \( a^3 + b^3 = (a+b)(a^2 - ab + b^2) \)

Think of the cube root as \(a\) and the other term as \(b\) and multiply by the trinomial factor from the right hand side above.

\[
\frac{1}{\sqrt[3]{6} + 5} = \frac{1}{\sqrt[3]{6} + 5} \cdot \frac{(\sqrt[3]{6})^2 - 5 \sqrt[3]{6} + 25}{(\sqrt[3]{6})^2 - 5 \sqrt[3]{6} + 25} = \frac{(\sqrt[3]{6})^2 - 5 \sqrt[3]{6} + 25}{6 + 5^3} = \frac{(\sqrt[3]{6})^2 - 5 \sqrt[3]{6} + 25}{121}
\]

**Complex Number \(a + bi\) in the Denominator**

Just treat \(i\) as a square root:

\[
\frac{2 - 3i}{3 + 4i} = \frac{2 - 3i}{3 + 4i} \cdot \frac{3 - 4i}{3 - 4i} = \frac{-6 - 17i}{25}
\]

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