1. Consider the following initial value problem.
   \[ \dot{y} = y^2 t \quad y(0) = 2 \]
   a. Are the conditions for existence, uniqueness met? Explain. State the conditions.
   b. Solve the initial value problem and indicate the solution curve on your sketch. State the domain of definition.

2. \[ \frac{dy}{dt} = y(1 - y) \]
   Sketch the solution curves, DE graph and phase line.
   Identify and classify all equilibrium solutions.
   Describe the long term behavior \((t \to \pm\infty)\) in all regions.

3. Solve the following differential equations. (You may need to leave your solution in implicit form or in terms of nonelementary integrals)
   \[
   \frac{dy}{dx} + xy = \sin x \\
   dx + (3xy^2 - y - 1)dy = 0 \\
   x^2dx + (2y^2 + xy)dy = 0 \\
   2ydx + (5y + 2x)dy = 0 \\
   \frac{dy}{dx} + xy = (\sin x) y^3
   \]

4. \[ \ddot{y} + b\dot{y} + 9y = \sin \omega t \quad y(0) = 0, \quad \dot{y}(0) = 0 \]
   For what values of \(b\) and \(\omega\) is the solution periodic?
   For what values are there frequency beats?
   Solve the system in the resonant case and sketch the solution.
   If \(b = 2\) describe the long term behavior of the system. YOU NEED NOT SOLVE.

5. Solve
   \[
   y'' + 3y' + 2y = t + \cos t + e^t \\
   x^2y'' + 4xy' + 3y = 0
   \]

6. Classify each of the following sets of functions as linearly dependent or linearly independent. Explain. You may use any of the theorems given in class as long as you state the theorem.
   \{\sin^2 t, \cos 2t, 5\}
   \{x, e^x, x + 2\}
   \{x, x + 1, x + 2\}
   \{\sin t, \sin 2t, \sin 3t\}
7. Solve
\[ x^2y'' + xy' + (x^2 - 25)y = 0 \]
\[ x^2y'' + xy' + (x^2 - 1/4)y = 0 \]
\[ x^2y'' + xy' - (x^2 + 25)y = 0 \]
\[ x^2y'' + xy' - (x^2 + 1/4)y = 0 \]
\[ x^2y'' + xy' - (5x^2 - 10)y = 0 \]

8. a. Find the eigenfunctions of the following boundary value problems (BVP's).
\[ y'' + \lambda y = 0 \quad y(0) = 0, y(L) = 0 \]
\[ y'' + \lambda y = 0 \quad y(0) = 0, y(L) = 0 \]
\[ y'' + \lambda y = 0 \quad y(0) = 0, y(L) = 0 \]

b. Write the orthogonality relation for the eigenfunctions of the following BVP's. Do not solve.
\[ y'' - y + \lambda y = 0 \quad y(0) = 0, y(L) = 0 \]
\[ xy'' + 3x^2y' + 5y + \lambda y = 0 \quad y(0) = 0, y(5) = 0 \]
\[ x^2y'' + 2x y' + 5y + \lambda xy = 0 \quad y(0) \text{ is finite, } y(1) + y'(1) = 0 \]

9. The temperature in a rod of length L with is determined from the BVP given below. Solve for \( u(x, t) \).
\[ k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad 0 < x < L, \quad t > 0 \]
\[ u(0, t) = u(L, t) = 0, \quad t > 0 \]
\[ u(x, 0) = 100, \quad 0 < x < L \]

10. \[ \ddot{y} + 9y = f(t) \quad y(0) = y_0, \quad \dot{y}(0) = 0 \]
a. Solve the differential equation with \( y_0 = 0 \) and \( f(t) = \delta(t - 1) \)
b. Solve the differential equation with \( y_0 = 0 \) and \( f(t) = H(t - 1) \)
c. Solve the differential equation with \( y_0 = 0 \) and \( f(t) = 0 \) for \( 0 \leq t < 1 \) and \( f(t) = t \) shifted forwards in time by 1 for \( t \geq 1 \). Sketch \( f(t) \) first.
d. Solve the differential equation with \( y_0 = 0 \) and \( f(t) \) is \( t \) for \( 0 \leq t < 1 \) and \( f(t + 1) = f(t) \). Sketch \( f(t) \) first.