Final for Calculus 2

The final will be ten questions. Below is a study guide for each of the ten questions.

1. Integration. (You will need to know substitution, integration by parts, partial fraction decomposition and trig substitution for sin's and cos's.)
   a. \[ \int x^4 e^{-x^2} \, dx \]
   
   b. Each integral can be evaluated with integration by parts. For each part state:
      1. What \( u \) is,  2. What \( dv \) is,  3. How many times integration by parts is needed.
      \[ \int x \ln x \, dx \]  \[ \int x^2 \ln x \, dx \]  \[ \int e^x \sin x \, dx \]  \[ \int x^3 e^x \, dx \]
   
   c. Derive \( \cos^2 x = \frac{1}{2} (1 + \cos 2x) \)

   d. \[ \int \sin^3 (5x) \, dx \]  \[ \int \sin^2 (5x) \cos^2 (5x) \, dx \]  \[ \int \sin^3 (5x) \cos^2 (5x) \, dx \]  \[ \int \sin^2 x \cos^3 x \, dx \]  \[ \int \cos^5 x \, dx \]

   e. \[ \int \frac{1}{u^2 + 2u + 5} \, du \]

   f. \[ \int \frac{1}{x(x + 1)} \, dx \]  \[ \int x e^{-x^2} \, dx \]  \[ \int \frac{\ln x}{x^2} \, dx \]  \[ \int \frac{1}{x^2 + 4x + 7} \, dx \]

2. Limits (You will need to know L'Hôpital's rule for indeterminate forms, especially when to take logs. Please review which forms are indeterminate.)
   a. \[ \lim_{x \to +\infty} x^2 e^{-2x} \]  \[ \lim_{x \to -\infty} x^2 e^{-2x} \]  \[ \lim_{x \to +\infty} xe^x \]  \[ \lim_{x \to -\infty} xe^{-x} \]  \[ \lim_{x \to +0^+} \frac{x}{\ln x} \]  \[ \lim_{x \to +0^-} \frac{x}{\ln x} \]  \[ \lim_{x \to +\infty} \frac{x}{\ln x} \]  \[ \lim_{x \to -1^+} \frac{x}{x - 1} \]  \[ \lim_{x \to -1^-} \frac{x}{x - 1} \]

   Which of the above are: 0⁺, 0⁻, +∞, −∞, 1⁺, 1⁻?

   b. Which of the below are not indeterminate forms?
      \[ \frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \frac{0}{\infty}, \quad \frac{\infty}{0}, \quad 1^\infty, \quad \infty^0, \quad 0^\infty, \quad 0^0, \quad 0^1, \quad 0^\infty, \quad 1^0, \quad \infty, \quad (0 - \infty), \quad (\infty - 0), \quad (\infty - \infty), \quad (0 \cdot \infty), \quad (1 \cdot \infty), \quad (\infty \cdot \infty) \]

   c. Evaluate
      \[ \lim_{x \to 0^+} (3x)^x \]  \[ \lim_{x \to +\infty} (\ln x)^{1/2} \]  \[ \lim_{x \to -1^+} \frac{e^x}{x} \]  \[ \lim_{x \to 0^+} \frac{\sin x}{x} \]  \[ \lim_{x \to +\infty} \sin x \]  \[ \lim_{x \to -\infty} \sqrt{x} \ln x \]  \[ \lim_{x \to +\infty} x \ln x \]  \[ \lim_{x \to -\infty} (\sin x)^{\ln x} \]  \[ \lim_{x \to 0^+} (1 + 3x)^x \]  \[ \lim_{x \to 1} (\ln x)^{\sin x} \]

   d. Prove \[ \lim_{x \to +\infty} \left( 1 + \frac{1}{x} \right)^{a/x} = \lim_{x \to +\infty} \left( 1 + \frac{a}{x} \right)^{x/b} = \lim_{x \to 0^+} (1 + ax)^{1/bx} = \lim_{x \to 0^+} (1 + x)^{a/bx} \]

   Evaluate the above limit. Hint: Squash your algebraic thoughts.
3. Improper integrals. (You will need to know how to recognize and use results for the $p$ integral. Don't forget the comparison test. Please remember that an improper integral can be convergent or divergent.)

   a. Consider $I = \int_{0}^{\infty} \frac{1}{x \ln x} \, dx$ and $J = \int_{0}^{\infty} \frac{\ln x}{x} \, dx$.

      a. Find whether each integral converges or diverges. Evaluate the integral if it converges.
      b. Explain your results in terms of $p$ integrals, algebraic and logarithmic growth.

   b. Is $\int_{0}^{\infty} e^{-x^2} \, dx$ convergent or divergent? Explain. Correct the following:

      $$\int_{0}^{\infty} e^{-x^2} \, dx = \int_{0}^{\infty} e^{-t^2} \, dt = -\frac{\sqrt{\pi}}{2} \quad \text{let } t = \frac{1}{x}$$

   c. Explain why each integral is improper, and test for convergence. Prove all results.

      $\int_{0}^{\infty} \frac{1}{x^2} \, dx$, $\int_{1}^{\infty} \sin x \, dx$ Hint: Use the definition 

      $\int_{0}^{\infty} \frac{1}{x^p} \, dx$ Hint: $\cos x$ is decreasing on $[0,1]$.

      $\int_{3}^{\infty} \frac{dx}{\sqrt{x^2 - 4}}$, $\int_{3}^{\infty} \frac{1}{\sqrt{x^2 + 7}} \, dx$, 

      $\int_{2}^{\infty} \frac{\ln x}{x^2} \, dx$, $\int_{0}^{\infty} \frac{1}{1 + x^2} \, dx$, 

      $\int_{0}^{\infty} e^{-1+x^2} \, dx$.

   d. $\int_{0}^{\infty} \frac{\sin x}{x} \, dx$ Explain why the integral is not improper.

   e. For each of the following integrals prove for what values of $p$, if any, it converges?

      a. $\int_{2}^{\infty} \frac{1}{(x-1)^p} \, dx$  b. $\int_{1}^{\infty} \frac{1}{(x-1)^p} \, dx$  c. $\int_{2}^{\infty} \frac{1}{(x-1)^p} \, dx$  d. $\int_{1}^{\infty} \frac{1}{(x-1)^p} \, dx$

   f. Consider the function $f(x) = \frac{1}{1 + x^2}$. Express the area above the x axis and under the graph of $f$ as an integral. Show that the integral is improper. Discuss convergence. Evaluate if possible.

4. Sequences (You will be expected to test for convergence, prove if a sequence is increasing or decreasing, and use the result that a sequence is convergent if it is monotone decreasing (increasing) and bounded below (above), resp.

   a. Prove whether each of the following sequences are (eventually) increasing, decreasing, or neither? State whether each sequence is convergent or divergent. Find the limit of each convergent sequence.

      $$\left\{ 1 - \frac{1}{n^2} \right\}, \left\{ \frac{s^n}{n!} \right\}, \left\{ \frac{n^2}{e^{3n}} \right\}, \left\{ \frac{3n+1}{5n-2} \right\}, \left\{ \sqrt[n]{n!} \right\}, \left\{ n! n^{-\ln n} \right\}, \left\{ \frac{e^{-\frac{1}{n^2}}}{n!} \right\}, \left\{ \frac{2^n}{n!} \right\}, \left\{ \frac{n!}{1+n^2} \right\}, \left\{ (-1)^n \ln n \right\}$$

   b. Consider the sequence $\left\{ \frac{(-1)^n \ln n}{n} \right\}$. Prove whether the sequence is convergent or divergent.

5. Evaluating infinite sums. (You will need to know how to find the formula for the sum of a geometric series)

   Find each sum. Leave all answers in exact form.

   $\sum_{n=3}^{\infty} (-1)^{n+1} \left( \frac{3}{5} \right)^n$, $\sum_{n=0}^{\infty} (x-5)^{3n}$, $\sum_{n=0}^{\infty} \left[ (\frac{2}{3})^n - (\frac{3}{5})^n \right]$, $\sum_{n=1}^{\infty} \left[ \frac{3^n}{n!} \right]$.

   $\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} e^{-n}$, $\sum_{n=0}^{\infty} \frac{(\frac{1}{2})^n}{2^n}$, $\sum_{k=5}^{\infty} \frac{3}{5} - \left( \frac{2}{3} \right)^k$.
6. Convergence of Series (There will be three parts: positive terms, alternating, and power series)

a. Test each series for convergence or divergence. Prove each result.

\[
\sum_{n=1}^{\infty} \frac{(n+3)^2}{n^3} \sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n^2}\right) \sum_{n=1}^{\infty} e^{-\frac{1}{n^2}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2} + 5} \sum_{n=1}^{\infty} \frac{n^n}{n!} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{n=1}^{\infty} \frac{1}{n^6}
\]

\[
\sum_{n=0}^{\infty} \frac{1}{n+2} \sum_{n=0}^{\infty} e^{-\frac{1}{n^2}} \sum_{n=0}^{\infty} \frac{1+n^2}{n!} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{n=0}^{\infty} \frac{n^2}{n!} \sum_{n=1}^{\infty} n^{-ln n} \sum_{n=1}^{\infty} \frac{ln n}{n!}
\]

b. Test each of the series for absolute or conditional convergence or divergence.

\[
\sum_{n=1}^{\infty} (-1)^n \frac{n!}{n^n} \sum_{n=2}^{\infty} \frac{(-1)^n n}{\ln n} \sum_{n=2}^{\infty} (-1)^n \ln\left(1 - \frac{1}{n^2}\right) \text{ Hint: } \ln\left(1 - \frac{1}{n^2}\right) < 0 \sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^4 + 2}}
\]

c. Find the interval and radius of convergence for each series. Examine all endpoints and indicate the type of convergence.

\[
x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{n=1}^{\infty} \frac{(-2)^n (x-3)^n}{n} \sum_{n=1}^{\infty} \frac{(-1)^n (3x-2)^n}{n} \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}
\]

7. Taylor series. (You will need to know how to find a Taylor series and Lagrange's form of the remainder and how to estimate the error.

a. Find Lagrange's form of the remainder, \( R_n(x) \), when \( x = -\frac{1}{2} \), for the Taylor Series of \( \frac{1}{1+x} \) about \( x = 0 \). Bound \( c \) in your formula.

b. Bound the above remainder \( R_n(-1/2) \) if the series is truncated at the tenth degree term. Is the estimate smaller or larger than the actual value?

c. Find the exact error for truncating \( \sum_{n=0}^{\infty} x^n \) after 3 terms when \( x = \sqrt{4} \).

d. Consider the function \( f(x) = \frac{1}{1+x^2} \). Find the Taylor series of \( f \) about \( x = 0 \), in closed form. What is the radius of convergence? Truncate the Taylor series of \( f \) after the first 3 nonzero terms, and bound the error in evaluating \( f\left(\frac{1}{2}\right) \), using an appropriate theorem. Find the actual error of the truncation of the MacLaurin series of \( f(x) \) at \( x = \frac{1}{3} \) after 3 terms. Show that the actual error is indeed within your error bounds. Use only exact rational arithmetic.

e. Find the Taylor series and interval of convergence of \( \sin x \) about \( x = \frac{\pi}{4} \).

8. Polar coordinates (You will need to know how to convert from rectangular to polar, sketch polar curves and find the area of a region enclosed by polar curves)

a. Sketch \( r = 1 + 2\cos\theta \) from \( \theta = 0 \) to \( \theta = 2\pi \). Put arrows on figures to denote direction of increasing \( \theta \). Find the \((r, \theta)\) coordinates of each \( x \) and \( y \) intercept. Find the area of the “inside loop.” Leave your answer in exact form.

b. Sketch \( r = -\cos(5\theta) \) from \( \theta = 0 \) to \( \theta = 2\pi \). Put arrows on figures to denote direction of increasing \( \theta \). Find the area of a leaf in the first quadrant. Leave your answer in exact form.
9. Parametric curves. (You will need to know how to find the parametric equations for the graph of a curve and how to find the equations for the tangent line and how to sketch the curve using calculus. You will also need to know how to find the length of a curve.)
   a. Sketch the curve \( x = 4\cos(t),\ y = 9\sin(t) \)
   b. Find the equation in \( x - y \) coordinates for the tangent line to the curve at \( t = \pi / 4 \).
   c. Set up the integral for the length of the curve in the first quadrant. The integral cannot be evaluated in terms of elementary functions.
   d. Find \( y'(x) \) and \( y''(x) \) for the curve in terms of the parameter \( t \).

10. Evaluate sum of infinite series. (You will need to recognize the Taylor series for the basic functions given in class and to use termwise differentiation and integration to find sums.)
   a. Sum the series. Leave your answer in exact form. \[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2n\pi^{2n-1}}{(2n-1)!} \]
      Hint: Consider an appropriate Taylor series first. Modify and then differentiate.
   b. Sum the series. Leave your answer in exact form. \[ 1 + 2\left(\frac{\pi}{4}\right) + 3\left(\frac{\pi}{4}\right)^2 + 4\left(\frac{\pi}{4}\right)^3 + \cdots \]