Calculus 2  TAYLOR AND MACLAURIN SERIES

On its interval of convergence the sum of the given power series $\sum_{n=0}^{\infty} c_n(x - a)^n$ is a continuous function $f(x)$ with derivatives of all orders. What about the other way around? Given a function $f(x)$ with derivatives of all orders on an interval $I$ can we express $f(x)$ as a sum of a power series on $I$? If yes, what are the coefficients? We start by assuming $f(x)$ to be the sum of a power series with positive radius of convergence.

$$f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots + c_n(x - a)^n + \cdots$$

Recall that a power series for $f(x)$ can be differentiated term-by-term and the resulting sum converges to $f'(x)$ within the interval of convergence. By repeated term-by-term differentiation within $I$

$$f'(x) = 1c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + 4c_4(x - a)^3 \cdots n c_n(x - a)^{n-1} + \cdots$$
$$f''(x) = 2 \cdot 1c_2 + 3 \cdot 2c_3(x - a) + 4 \cdot 3c_4(x - a)^2 + \cdots + n(n - 1)c_n(x - a)^{n-2} + \cdots$$
$$f'''(x) = 3 \cdot 2 \cdot 1c_3 + 4 \cdot 3 \cdot 2c_4(x - a) + \cdots + n(n - 1)(n - 2)c_n(x - a)^{n-3} + \cdots$$
$$\vdots$$
$$f^{(n)}(x) = n! \cdot c_n + \text{sum of terms with factor } (x - a)$$

since these equations hold for $x = a$, we have,

$$f'(a) = 1c_1$$
$$f''(a) = 2 \cdot 1c_2$$
$$f'''(a) = 3 \cdot 2 \cdot 1c_3$$
$$\vdots$$
$$f^{(n)}(a) = n! \cdot c_n$$

We have just shown that if a power series $\sum_{n=0}^{\infty} c_n(x - a)^n$ exists that converges to the values of $f(x)$ on $I$ then the power series is unique and its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

The power series for $f(x)$ is the unique series

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \cdots$$

The question remains: Given an $f(x)$ infinitely differentiable on $I$, if we generate the power series on the RHS, does it converge? Does it converge to $f(x)$ on $I$? We call the series on the RHS the Taylor Series of $f$ about $x = a$. If $a = 0$ the Taylor Series is called the MacLaurin Series

**Definition**: Taylor Series and MacLaurin Series

Let $f$ be a function with derivatives of all orders in some interval containing $a$ inside the interval. The Taylor Series of $f(x)$ about $x = a$ (generated by $f(x)$ at $x = a$) (representation of $f(x)$ about $x = a$) is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x - a)^k = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \cdots$$

The MacLaurin Series is the Taylor series of $f(x)$ about $x = 0$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$
Recall that the linearization of \( f(x) \) at \( x = a \) is the polynomial \( P_1(x) = f(a) + f'(a)(x - a) \) (equation of tangent line). This is the best approximation of \( f(x) \) in the neighborhood of \( x = a \). Similarly if \( f(x) \) has higher order derivatives in the neighborhood of \( x = a \) then it has higher order polynomial approximations. These polynomials which match the value of \( f \) and its higher order derivatives at \( x = a \) are called the Taylor polynomials of \( f \). It turns out the higher order Taylor polynomials are the best polynomial approximations of their respective degrees.

**Definition: Taylor polynomial of order \( n \)**

If \( f(x) \) has derivatives of all orders up to \( n \) in some interval \( I \) containing \( a \) then the polynomial

\[
P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!} (x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!} (x - a)^n = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k
\]

is called the \( n^{th} \) order Taylor polynomial for \( f(x) \) about \( x = a \).

If the \( n^{th} \) order derivative of \( f(x) \) is zero then the \( n^{th} \) order Taylor polynomial, \( P_n(x) \), is not of degree \( n \). For instance the first order Taylor polynomial of \( \cos x \) about \( x = 0 \) has degree zero not 1 because the first derivative of \( \cos x \) is zero at \( x = 0 \). \( P_0(x) = 1 \quad P_1(x) = 1 + 0x = 1 \).

The Taylor Series is a power series about \( x = a \). Therefore there are only three possibilities for the \( x \) interval of convergence of the series.

1. The series converges absolutely for all \( x \). The radius of convergence is \( \infty \).
2. The series converges absolutely only for \( x = a \). The radius of convergence is zero.
3. The series converges absolutely in an interval about \( x = a \), \((a - R, a + R)\) and diverges outside the interval. The endpoints need to be tested separately. \( R \) is called the radius of convergence. \( 0 < R < \infty \)

For any given Taylor Series for \( f(x) \) we need to determine the following:

1. What is the interval of convergence for \( x \)? We use the ratio test.
2. Does the series converge to \( f(x) \)?

If \( a = 0 \) the Taylor polynomial is called the MacLaurin polynomial.

**Example:** \( f(x) = e^x \). Find the \( 3^{rd} \) order MacLaurin polynomial, \( P_3(x) \). This is a polynomial of degree \( \leq 3 \) for \( e^x \) about \( x = 0 \) which matches the value of \( e^x \) and the value of the first and second and third derivatives of \( e^x \) at \( x = 0 \) with the values of the polynomial at \( x = 0 \).

\[
P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3
\]

\[
f(0) = e^0 = 1, f'(0) = x^0 = 1, f''(0) = e^0 = 1, f'''(0) = e^0 = 1
\]

\[
P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}
\]
\( P_3(x) \) is the only polynomial of degree \( \leq 3 \) with error, \( E(x) = e^x - P_3(x) \), zero at \( x = 0 \) and error negligible compared to \( x^3 \). \( \lim_{x \to 0} \frac{E(x)}{x^3} = 0 \).

**The Taylor polynomial \( P_n(x) \) for \( f(x) \) is the only polynomial of degree \( \leq n \) with error both zero at \( x = a \) and negligible compared to \( (x-a)^n \).**

**Error** \( E(x) = f(x) - P_n(x) \)

\[
\lim_{x \to a} \frac{E(x)}{(x-a)^n} = 0
\]

Find the MacLaurin series of \( e^x \)

\[
f(0) = f'(0) = f''(0) = \cdots f^{(n)}(0) = 1. \text{ The MacLaurin series of } e^x \text{ is } 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}
\]

Does the series converge? \( \lim_{n \to \infty} \frac{x^{n+1}n!}{(n+1)x^n} = \lim_{n \to \infty} \frac{x}{n+1} = 0 < 1 \). The series converges absolutely for all \( x \) by the ratio test. The question remains, does the Taylor series converge to \( f(x) \)?

Example: Find the Taylor series for \( \sin x \) about \( x = 0 \)

The Taylor series for \( f(x) \) about \( x = 0 \) is \( \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \)

\[
f(0) = \sin x \bigg|_{x=0} = 0
f'(0) = \cos x \bigg|_{x=0} = 1
f''(0) = -\sin x \bigg|_{x=0} = 0
f'''(0) = -\cos x \bigg|_{x=0} = -1
f^{(4)}(0) = \sin x \bigg|_{x=0} = 0
f^{(5)}(0) = \cos x \bigg|_{x=0} = 1
\]

The Taylor series for \( \sin x \) about \( x = 0 \) is

\[ 0 + 1x + \frac{0x^2}{2!} - \frac{1x^3}{3!} + \frac{0x^4}{4!} + \frac{1x^5}{5!} + \frac{0x^6}{6!} - \frac{1x^7}{7!} + \cdots = \]

\[ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \]. Verify this series converges absolutely for all \( x \).
Example: Find the MacLaurin series of $\cos x$

The Taylor series for $f(x)$ about $x = 0$ is $\sum_{n=0}^{\infty} f^{(n)}(0) \frac{x^n}{n!}$

$f(0) = \cos x_{x=0} = 1$
$f'(0) = -\sin x_{x=0} = 0$
$f''(0) = -\cos x_{x=0} = -1$
$f'''(0) = \sin x_{x=0} = 0$
$f^{(4)}(0) = \cos x_{x=0} = 1$
$f^{(5)}(0) = -\sin x_{x=0} = 0$

The Taylor series for $\cos x$ about $x = 0$ is $1 + 0x - \frac{1x^2}{2!} + \frac{0x^3}{3!} + \frac{1x^4}{4!} - \frac{0x^5}{5!} - \frac{1x^6}{6!} + \frac{0x^7}{7!} + \cdots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$. Verify this series converges absolutely for all $x$.

Example: Find the Taylor series of $\sin 2x$ about $x = 0$

The Taylor series for $f(x)$ about $x = 0$ is $\sum_{n=0}^{\infty} f^{(n)}(0) \frac{x^n}{n!}$

$f(0) = \sin 2x_{x=0} = 0$
$f'(0) = 2 \cos 2x_{x=0} = 2$
$f''(0) = -2^2 \sin 2x_{x=0} = 0$
$f'''(0) = -2^3 \cos 2x_{x=0} = -2^3$
$f^{(4)}(0) = 2^4 \sin 2x_{x=0} = 0$
$f^{(5)}(0) = 2^5 \cos 2x_{x=0} = 2^5$

In order to recognize powers and factorials for formulas it is useful not to multiply out coefficients.

The Taylor series for $\sin 2x$ about $x = 0$ is $0 + 2x + \frac{0x^2}{2!} - \frac{2^2 x^3}{3!} + \frac{0x^4}{4!} + \frac{2^4 x^5}{5!} + \frac{0x^6}{6!} - \frac{2^7 x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}$

Note that we could have find the Taylor Series directly by substitution.

Example: Find the Taylor series for $\ln x$ about $x = 1$.

The Taylor Series of $f(x)$ about $x = 1$ is $\sum_{n=0}^{\infty} f^{(n)}(1) \frac{(x-1)^n}{n!}$

$f(1) = \ln x_{x=1} = 0$
$f'(1) = \frac{1}{x_{x=1}} = 1$
Keep going until you recognize a pattern and can generate a formula for the $n$th derivative of $f$ with respect to $x$ evaluated at $x = 1$, $f^{(n)}(1) = (-1)^{n+1} (n-1)! \ n \geq 1$

The Taylor series for $\ln x$ about $x = 1$ is

$$
\sum_{n=1}^{\infty} \frac{(-1)^{n+1} (n-1)!}{n!} (x-1)^n = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n} .
$$

This series converges absolutely for $|x-1| < 1$, diverges at $x = 1$ and converges conditionally at $x = 2$.

Example: Find the Taylor series of $f(x) = x^3 - 3x^2 + 2x + 1$ about $x = 0$

$$
f(0) = 1 \\
f'(0) = 3x^2 - 6x + 2, \ x=0 = 2 \\
f''(0) = 6x - 6, \ x=0 = -6 \\
f'''(0) = 6 \\
f^{(n)}(0) = 0 \ \ \ n \geq 4
$$

The Taylor series of $f(x)$ is $1 + 2x - \frac{6x^2}{2!} + \frac{-6x^3}{3!} = 1 + 2x - 3x^2 + x^3$ of course!!

If a power series of $x$ converges to $f(x)$ on some open interval about $x = 0$ then the power series is the Taylor series.

You will need to memorize the following Taylor series about $x = 0$ (MacLaurin series). The interval of convergence is also given.

$$
e^x \quad 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (-\infty, \infty)$$

$$\sin x \quad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad (-\infty, \infty)$$

$$\cos x \quad 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad (-\infty, \infty)$$

$$\ln(1+x) \quad x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \quad (-1, 1]$$

$$\frac{1}{1+x} \quad 1 - x + x^2 - x^3 + x^4 - \cdots = \sum_{n=0}^{\infty} (-1)^n x^n \quad (-1, 1)$$

For homework problems see [http://voyager.dvc.edu/~lmonth/Calc2/Hw4Series.pdf](http://voyager.dvc.edu/~lmonth/Calc2/Hw4Series.pdf)