Calculus 2  TAYLOR SERIES CONVERGENCE AND TAYLOR REMAINDER

Let the difference between $f(x)$ and its Taylor polynomial approximation of order $n$ be $R_n(x)$.

$$f(x) = P_n(x) + R_n(x)$$

Consider $R_n(x)$ to be the remainder with $f(x)$ the exact value and $P_n(x)$ the approximate value.

$|R_n(x)|$ is the error of the approximation of $f(x)$ by its Taylor polynomial of order $n$. We need to estimate $|R_n(x)|$.

Taylor's Theorem which is a generalization of the Mean Value Theorem gives a formula for the remainder.

Taylor's theorem
If $f$ is differentiable through order $n + 1$ in some open interval containing $a$ then for each $x$ in $I$ there exists a number $c$ between $x$ and $a$ such that

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where $R_n$, the remainder, is given in the form

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

where $c$ lies between $x$ and $a$.

If for all $x$ in $I$ we have $\lim_{n \to \infty} R_n(x) = 0$ then the Taylor Series of $f(x)$ about $x = a$ converges to $f(x)$ on $I$ and we write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

If the absolute value of the $n^{th}$ derivative of $f(x)$ is bounded above by a constant then Taylor’s Inequality provides an easier bound for Taylor’s remainder.

Taylor’s Inequality:
If all the conditions of Taylor's theorem are satisfied and in addition there is a positive constant $K$ such that,

$$|f^{(n+1)}(c)| \leq K$$

for all $c$ between $a$ and $x$ then Taylor’s remainder satisfies the inequality

$$|R_n(x)| \leq \frac{K}{(n+1)!}|(x-a)|^{n+1}$$

If these condition hold for every $n$ then the series converges $f(x)$. (By the squeeze theorem. See below.)

Finally recall we have a bound for the remainder of an alternating series which satisfies the conditions of the Alternating Series Test. The truncation error, $S - S_n$, has magnitude less than the magnitude of the next term in the series. If the Taylor Series of $f(x)$ about $x = a$ converges to $f(x)$ in some interval $I$ containing $a$ and in addition the Taylor series of $f(x)$ about $x = a$ is an alternating series for some $x^*$ in the interval of convergence and the alternating series satisfies the conditions of the Alternating Series Test then the truncation error is less than the absolute value of the next term in the Taylor series and the sign of the error is the sign of the next term. In this case we know the sign of the error which Taylor's inequality does not provide.

Example: Find the interval of $x$ convergence where the Taylor Series for $\sin x$ about $x = 0$ converges to $\sin x$.

The Taylor series for $\sin x$ about $x = 0$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

which converges absolutely for $(-\infty, \infty)$. We now find where it converges to $\sin x$. The series has only odd-powered terms and for $n = 2k + 1$ Taylor's theorem gives

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + R_{2k+1}(x)$$
All the derivatives of $\sin x$ have absolute value less than or equal to 1. By Taylor's Inequality

$$0 \leq |R_{2k+1}(x)| \leq \frac{|x|^{2k+2}}{(2k+2)!}.$$ 

$$\lim_{k \to \infty} \frac{|x|^{2k+2}}{(2k+2)!} = 0.$$ By the Squeezing theorem $\lim_{k \to \infty} |R_{2k+2}(x)| = 0$ for every $x$ and the MacLaurin series for $\sin x$ converges to $\sin x$ for every $x$.

We have just shown the Taylor series for $\sin x$ about $x = 0$ converges to $\sin x$ on $(-\infty, \infty)$ by showing that $\lim_{n \to \infty} R_n(x) = 0$ on $(-\infty, \infty)$. Similarly for $\cos x$. Finally we can write

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad (-\infty, \infty)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad (-\infty, \infty)$$

Note that the Taylor Series for $\sin x$ and $\cos x$ are alternating series for $x \neq 0$.

Now let's examine the Taylor Series for $e^x$. The MacLaurin series for $e^x$, $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, converges for $(-\infty, \infty)$. The Taylor Series for $e^x$ is an alternating series for $x < 0$. By Taylor's theorem.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + R_n$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} = \frac{e^c x^{n+1}}{(n+1)!}$$

where $c$ is between $x$ and 0.

$$\lim_{n \to \infty} |R_n(x)| = e^x \lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} \quad c \text{ is between } x \text{ and } 0.$$ 

Case 1. $x < 0$, $x < c < 0$, $e^c < 1$

$$0 < |R_n(x)| = e^c \frac{|x|^{n+1}}{(n+1)!} < \frac{|x|^{n+1}}{(n+1)!}$$

$$\lim_{n \to \infty} R_n(x) = 0 \quad x < 0$$

Case 2. $x > 0$, $0 < c < x$, $e^c < e^x$

$$0 < |R_n(x)| = e^c \frac{|x|^{n+1}}{(n+1)!} < e^x \frac{|x|^{n+1}}{(n+1)!}$$

$$\lim_{n \to \infty} R_n(x) = 0 \quad x > 0$$

We have just shown the Taylor series of $e^x$ about $x = 0$ converges to $e^x$ on $(-\infty, \infty)$ because $\lim_{n \to \infty} R_n(x) = 0$ for all $x$. 


We never need to check $x = a$ because the Taylor series of $f(x)$ about $x = a$ is a power series in $x - a$ which by definition = $f(a)$ at $x = a$.

Example: Find Taylor's remainder for the Taylor series of $\sqrt{x}$ with $a = 4$ and $n = 3$. This means find the truncation error if we truncate the Taylor series of $\sqrt{x}$ about $x = 4$ at the term of order 3. (The third Taylor polynomial)

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1} \text{ c between x and a}$$

$$R_3(x) = \frac{f^{(3+1)}(c)}{(3+1)!} (x-4)^{3+1} = \frac{f^{(4)}(c)}{4!} (x-4)^4 = -\frac{15}{16} c^{-7/2} (x-4)^4 = -\frac{5}{128} c^{-7/2} (x-4)^4$$

$$R_3(x) = -\frac{5}{128} c^{-7/2} (x-4)^4 \text{ c between x and 4}.$$  

The calculations for the derivatives of $f(x)$ are shown below.

$$f(x) = \sqrt{x} = x^{1/2}$$
$$f'(x) = \frac{1}{2} x^{-1/2}$$
$$f''(x) = -\frac{1}{4} x^{-3/2}$$
$$f'''(x) = \frac{3}{8} x^{-5/2}$$
$$f^{(4)}(x) = -\frac{15}{16} x^{-7/2}$$

Example: Find Taylor's remainder for the Taylor series of $\frac{1}{x+1}$ with $a = 0$ and $n = 3$. This means find the truncation error if we truncate the Taylor series of $\frac{1}{x+1}$ about $x = 0$ at the third Taylor polynomial.

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1} \text{ c between x and a}$$

$$R_3(x) = \frac{f^{(3+1)}(c)}{(3+1)!} \left(\frac{1}{x+1}\right)^{3+1} = \frac{f^{(4)}(c)}{4!} \left(\frac{1}{x+1}\right)^4 = \frac{4!}{(c+1)^5} \left(\frac{1}{x+1}\right)^4 = \frac{x^4}{(c+1)^5}$$

$$R_3(x) = \frac{x^4}{(c+1)^5} \text{ c between x and 0}.$$  

The calculations for the derivatives of $f(x)$ are shown below.

$$f(x) = \frac{1}{x+1} = (x+1)^{-1}$$
$$f'(x) = -(x+1)^{-2}$$
$$f''(x) = 2(x+1)^{-3}$$
$$f'''(x) = -3 \cdot 2(x+1)^{-4}$$
$$f^{(4)}(x) = 4!(x+1)^{-5} = \frac{4!}{(x+1)^5}$$
Example: Find Taylor’s remainder for the MacLaurin series of $\frac{1}{1-x}$

\[ R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (1-x)^{n+1} = \frac{(n+1)!}{(1-x)^{n+2}} \frac{x^{n+1}}{(1-c)^{n+2}}(n+1)! = \frac{x^{n+1}}{(1-c)^{n+2}} \]

\[ R_n(x) = \frac{x^{n+1}}{(1-c)^{n+2}} \quad c \text{ between } x \text{ and } 0 \]

The calculations for the derivatives of $f(x)$ are shown below.

\[ f(x) = \frac{1}{1-x} = (1-x)^{-1} \]
\[ f'(x) = -(1-x)^{-2} (-1) = (1-x)^{-2} \]
\[ f''(x) = -2(1-x)^{-3} (-1) = 2(1-x)^{-3} \]
\[ f'''(x) = -3 \cdot 2(1-x)^{-4} (-1) = 3 \cdot 2(1-x)^{-4} \]
\[ f^{(4)}(x) = 4! (1-x)^{-5} = \frac{4!}{(1-x)^5} \]
\[ f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}} \]
\[ f^{(n+1)}(x) = \frac{(n+1)!}{(1-x)^{n+2}} \]

Example: Prove $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2$

\[ \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \text{ on } (-1, 1]. \]

Note the "=" sign. This is because the RHS is the Taylor series which converges to the function on $(-1, 1]$. \[ \lim_{n \to \infty} R_n(x) = 0 \text{ in this interval. To show } 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} \]

converges on $(-1, 1]$ use the ratio test. The series converges absolutely when \[ \lim_{n \to \infty} \left| \frac{x^{n+1}(n)}{(n+1)(x^n)} \right| = |x| < 1. \]

The radius of convergence is 1. The interval of convergence for $x$ is $(-1, 1)$. We need to test the endpoints separately. At \[ x = 1, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \quad \text{which converges conditionally. At } x = -1, \quad \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (-1)^n}{n} \]

\[ = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n} \] which diverges. \[ \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \text{ on } (-1, 1]. \]

At $x = 1$, \ \ln 2 = \[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \]
Example: Evaluate \( \ln 2 \) using a series which converges faster than \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \)

Replace \( x \) by \( -x \) in the Taylor series for \( \ln(1 + x) \)

\[
\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{on} \quad -1 < -x \leq 1
\]

\[
\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{on} \quad -1 \leq x < 1
\]

\[- \ln(1 - x) = \ln \frac{1}{1-x} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{on} \quad [-1, 1). \text{ At } x = 1/2 ,
\]

\[
\ln 2 = \frac{1}{2} + \left( \frac{1}{2} \right)^2 \frac{1}{2} + \left( \frac{1}{2} \right)^3 \frac{1}{3} + \left( \frac{1}{2} \right)^4 \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n}
\]

For homework see

http://voyager.dvc.edu/~lmonth/Calc2/Hw5Series.pdf
http://voyager.dvc.edu/~lmonth/Calc2/Hw6Series.pdf