Music and Fourier Series

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Thank you to my guinea pigs Janet, Niki, Taylor, Skyler and Pat.

These slides are available via my homepage: http://voyager.dvc.edu/~kedwards

Please feel free to ask questions throughout the talk. I may or may not know the answer, but I look forward to finding the limits of my knowledge so that I can learn more.
Idea #1: Sounds are caused by varying air pressure

Demos at:

http://www.iknowthat.com/ScienceIllustrations/sound/science_desk.swf

http://illuminations.nctm.org/Activity.aspx?id=4202
Idea # 2: These variations in air pressure can be represented in a graph

(from the first link in the previous slide)
A sinusoidal wave is a wave shaped like a sine wave or cosine wave, i.e. whose graph has the form
\[ f(x) = a\sin(bx+c) + d \]

From [http://illuminations.nctm.org/Activity.aspx?id=3589](http://illuminations.nctm.org/Activity.aspx?id=3589)
Hearing a sine wave

- Second tab on http://www.iknowthat.com/ScienceIllustrations/sound/science_desk.swf

- How does volume affect the shape of the graph? (Try “strike gently” option)

- How does pitch affect the shape of the graph? (Try different glasses)
Answers: Higher volume corresponds to higher amplitude. Higher pitches correspond to higher frequencies, thus lower wavelengths.

Next: Sing subsets of a major scale. Why does a scale “stop” after 8 notes? Why not earlier or later?
Idea#4: An octave has twice the frequency and half the wavelength
Idea#5: Strings on an instrument have a series of natural overtones.

Pass out “stringed instruments”

Play strings, hold at ½ point, 1/3 point, etc.

Source:
http://en.wikipedia.org/wiki/Fundamental_frequency
Side topic: Pairs of notes whose wavelengths have a simple ratio (2:1, 3:1, 3:2) sound pleasing together

- Sing major scale again, with drone on fundamental
- 2:1 gives ??
- 3:2 gives ??
- 4:3 gives ??
- 5:4 gives ??
Sing major scale again, with drone on fundamental.

- 2:1 gives octave
- 3:2 gives fifth
- 4:3 gives fourth
- 5:4 gives third

Show website with chart of consonances:
http://physics.info/music/
Side topic: Musical temperament

Ratios to the Tonic in a Diatonic Scale with Just Intonation
(C = 256 Hz Scientific Scale)

Source: http://physics.info/music/
Side topic: Musical temperament

Ratios of the Chromatic Intervals in a Scale with Equal Temperament

(A = 440 Hz American Standard Scale)

Also see http://mathforum.org/library/drmath/view/52470.html
Back to single strings and overtones...

Question: How do we graph a note from an instrument, which is more than just a pure tone?

Remember, we are now looking at the fundamental + overtones on a **single** note from an instrument, as opposed to comparing the fundamentals of two different notes.
Answer: We add the graphs

Go to the Desmos graphing calculator at
https://www.desmos.com/calculator

and graph the following:

- $f(x) = \sin x$
- $g(x) = \frac{1}{2} \sin 3x$
- $h(x) = \frac{1}{10} \sin 6x$
- $f(x) + g(x) + h(x)$
A note about periodic functions

Notice that when you add these sine waves, the result is not a sine wave, but it is still a periodic function.

A periodic function is a function that repeats itself exactly after a certain period $T$. Mathematically, we say:

**Defn.** $f(x)$ is periodic with period $T$ if $f(x) = f(x+nT)$ for all integers $n$. 
Examples of periodic functions

**Exs.** Sin(x) is periodic with period 2\(\pi\).
\[\frac{1}{2} \sin(3x)\] is periodic with period \(\frac{2}{3} \pi\).

You can also say that \(\frac{1}{2} \sin(3x)\) is periodic with period \(2\pi\), but in this case \(2\pi\) is not the smallest possible period. The smallest possible period is sometimes called “the period” or the “least period”.

Frequency is the inverse of period.

**Defn.** If $f(x)$ is periodic with least period $T$, then the frequency of $f(x)$ is $1/T$.

**Exs.** $\sin(x)$ has frequency $1/(2\pi)$
$1/2 \sin(3x)$ has frequency $3/(2\pi)$
The most common unit for frequency in music is Hz (Hertz).

**Defn.** Hz = 1/(sec)

**Ex.** The note A above middle C is commonly tuned to 440Hz. (This is known as A440 tuning.) A pure tone A at 440Hz is simply a sine wave vibrating 440 times per second.

Periods can be given in seconds, but milliseconds (ms) are common, since e.g. a note with frequency 440Hz has a period of $\frac{1}{440}$ sec = 0.002272 sec = 2.27 ms (approx.)
Graphs of Real Instruments

Flute playing A4 (440 Hz):

Saxophone playing C4 (262 Hz):

Source: http://hyperphysics.phy-astr.gsu.edu/hbase/music/musinscon.html
Idea #6: Different instruments playing the same note sound different because of their overtones.

Trombone playing Bflat3 (233 Hz):

Clarinet playing Bflat3 (233 Hz):
And, the note on a real instrument changes with time

(Show Garage Band. Sing into it and then zoom into the waveform in editing mode.)
Fourier Series

What are Fourier Series?

• Fourier Series are the mathematical way of writing what we have been talking about.
• A Fourier series takes a periodic function and expresses it as a (possibly infinite) sum of sinusoidal functions, plus a constant term.

\[ f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx), \]

i.e. \( f(x) = \frac{1}{2} a_0 + a_1 \cos(x) + a_2 \cos(2x) + a_3 \cos(3x) + \ldots \)
\[ + b_1 \sin(x) + b_2 \sin(2x) + b_3 \sin(3x) + \ldots \]
Question: Do all periodic functions have Fourier series?

• EVERY DIFFERENTIABLE PERIODIC FUNCTION can be rewritten this way, i.e. it has a Fourier series expansion.

• If $f$ is differentiable except for a jump discontinuity, it will still have an associated Fourier series expansion, which converges to $f$-bar. $f$-bar equals $f$, except that at the jump discontinuity its value is equal to the average of the left and right limits.

• Beyond this, convergence issues with Fourier series can be very hard.
Examples of Fourier Series

Source: http://mathworld.wolfram.com/FourierSeries.html
The first four partial sums of the Fourier series for a square wave.

Source:  
http://en.wikipedia.org/wiki/Fourier_series
The Million Dollar Question:

(Math) Given a periodic function, how do we find the sinusoidal functions that comprise it?

(Music) Given a constant sound, how do we find the pure tones that comprise it?
Spectrum Analysis

Flute Waveform

Harmonic | Amplitude
---|---
1 | 1.00
2 | 0.77
3 | 0.18
4 | 0.11
5 | 0.05
6 | 0.02
7 | 0.00
8 | 0.00
9 | 0.00

Higher in the range of the flute, the energy is mostly concentrated in the fundamental and second harmonic. For very high notes, it is almost a sine wave.

Flute tone by Jessica Bean, 4/29/97
More examples of spectrum analysis

See:

* [http://hyperphysics.phy-astr.gsu.edu/hbase/music/flutew.html](http://hyperphysics.phy-astr.gsu.edu/hbase/music/flutew.html)
* [http://hyperphysics.phy-astr.gsu.edu/hbase/music/tromw.html](http://hyperphysics.phy-astr.gsu.edu/hbase/music/tromw.html)
* [http://hyperphysics.phy-astr.gsu.edu/hbase/music/clarw.html](http://hyperphysics.phy-astr.gsu.edu/hbase/music/clarw.html)
* [http://hyperphysics.phy-astr.gsu.edu/hbase/music/vwav1.html](http://hyperphysics.phy-astr.gsu.edu/hbase/music/vwav1.html)

In the old days when people had stereos with components, you saw something like this all the time...where?
Spectrum analyzer

- Google image search for “stereo spectrum analyzer”
  https://www.google.com/search?q=stereo+spectrum+analyzer&tbm=isch

- Wikipedia page on Fourier series (definition animation)
  http://en.wikipedia.org/wiki/Fourier_series#Definition

- iSpectrum program
Now, how is this done mathematically? The key to the puzzle is the following five integrals. Try one of them!

\[
\begin{align*}
\int_{-\pi}^{\pi} \sin(mx) \sin(nx) \, dx \\
\int_{-\pi}^{\pi} \cos(mx) \cos(nx) \, dx \\
\int_{-\pi}^{\pi} \sin(mx) \cos(nx) \, dx \\
\int_{-\pi}^{\pi} \sin(mx) \, dx \\
\int_{-\pi}^{\pi} \cos(mx) \, dx
\end{align*}
\]
Answers to integrals

In other words, if you integrate over a common period,

• Integrating a sin function times itself is NOT zero
• Integrating a sin function times a sin of any other frequency, a cos of any frequency, or a constant gives ZERO.

(Similarly for cosine)

where \( \delta_{mn} = \begin{cases} 
0 & \text{if } m \neq n \\
1 & \text{if } m = n 
\end{cases} \)
Finding a Coefficient

Suppose a periodic function $f(x)$ has a Fourier series and we want to find the coefficient of the $\sin(2x)$ term.

Again, we know the Fourier series is:

$$f(x) = \frac{1}{2} a_0 + a_1 \cos(x) + a_2 \cos(2x) + a_3 \cos(3x) + ...$$

$$+ b_1 \sin(x) + b_2 \sin(2x) + b_3 \sin(3x) + ....$$

We just don’t know the value of $b_2$ (or any other coefficient).

The key idea is to integrate $f(x)$ multiplied by the term we are interested in.
\[ \int_{-\pi}^{\pi} \sin(2x)f(x) \, dx = \int_{-\pi}^{\pi} \left[ \frac{1}{2} a_0 + a_1 \cos(x) + a_2 \cos(2x) + a_3 \cos(3x) \ldots \right. \\
\left. \quad + b_1 \sin(x) + b_2 \sin(2x) + b_3 \sin(3x) + \ldots \right] \, dx \]

= (distribute \sin(2x))

= (rewrite as sum of integrals)

= (pull out constants)
Integration cont.

\[\begin{align*}
&= \frac{1}{2} a_0 \int_{-\pi}^{\pi} \sin(2x) dx \\
&+ a_1 \int_{-\pi}^{\pi} \sin(2x) \cos(x) dx + a_2 \int_{-\pi}^{\pi} \sin(2x) \cos(2x) dx + a_3 \int_{-\pi}^{\pi} \sin(2x) \cos(3x) dx + \ldots \\
&+ b_1 \int_{-\pi}^{\pi} \sin(2x) \sin(x) dx + b_2 \int_{-\pi}^{\pi} \sin(2x) \sin(2x) dx + b_3 \int_{-\pi}^{\pi} \sin(2x) \sin(3x) dx + \ldots
\end{align*}\]
Integration cont.

\[
= \frac{1}{2} a_0 \int_{-\pi}^{\pi} \sin(2x)dx
\]

\[
+ a_1 \int_{-\pi}^{\pi} \sin(2x) \cos(x)dx + a_2 \int_{-\pi}^{\pi} \sin(2x) \cos(2x)dx + a_3 \int_{-\pi}^{\pi} \sin(2x) \cos(3x)dx + \ldots
\]

\[
+ b_1 \int_{-\pi}^{\pi} \sin(2x) \sin(x)dx + b_2 \int_{-\pi}^{\pi} \sin(2x) \sin(2x)dx + b_3 \int_{-\pi}^{\pi} \sin(2x) \sin(3x)dx + \ldots
\]

\[
= 0 + 0 + 0 + \ldots + 0 + b_2 \pi + 0 + \ldots
\]
\[ \frac{1}{2} a_0 \int_{-\pi}^{\pi} \sin(2x)dx \]

\[ + a_1 \int_{-\pi}^{\pi} \sin(2x) \cos(x)dx + a_2 \int_{-\pi}^{\pi} \sin(2x) \cos(2x)dx + a_3 \int_{-\pi}^{\pi} \sin(2x) \cos(3x)dx + \ldots \]

\[ + b_1 \int_{-\pi}^{\pi} \sin(2x) \sin(x)dx + b_2 \int_{-\pi}^{\pi} \sin(2x) \sin(2x)dx + b_3 \int_{-\pi}^{\pi} \sin(2x) \sin(3x)dx + \ldots \]

\[ = 0 + 0 + 0 + 0 + \ldots + 0 + b_2 \pi + 0 + \ldots \]

Thus \( b_2 = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(2x) f(x) dx \)
Formulas for Fourier Coefficients

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx)f(x)\,dx \]
\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx)f(x)\,dx \]
Formulas for Fourier Coefficients

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\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx) f(x) \, dx \]

Note: When \( n=0 \), the first formula gives \( a_0 = 2(\text{avg value of } f(x)) \). Thus we write the constant term of the Fourier series as \( \frac{1}{2} a_0 \).
Formulas for Fourier Coefficients with functions of other periods

Period $2\pi$:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx) f(x) \, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx) f(x) \, dx$$

Period $P$:

$$a_n = \frac{2}{P} \int_{x_0}^{x_0+P} \cos\left(\frac{2\pi nx}{P}\right) f(x) \, dx$$

$$b_n = \frac{2}{P} \int_{x_0}^{x_0+P} \sin\left(\frac{2\pi nx}{P}\right) f(x) \, dx$$
In Reality

Of course with a generic periodic function we don’t know how to solve the integral with the exact closed-form techniques you learn in Calculus II. So instead we just approximate, e.g., using a Riemann sum.

A spectrum analyzer is doing this constantly for many frequencies at once!! (on iSpectrum, point out the button for “extra CPU”.)
So what do other periodic functions sound like?

You can try and create your own waveforms here: http://library.thinkquest.org/19537/java/Wave.html

It’s hard to draw nice sine waves! This applet makes it easier; all you do is choose your Fourier coefficients: http://web.mit.edu/jorloff/www/jmoapplets/fouriersound3/fouriersound.html
Thinking Questions (some of these I know the answers to, some I don’t):

• Will any periodic function \( f(x) \) produce a sound?
• Why is there no sound in space?
• If all the strings on a guitar are the same length, don’t the notes all have the same wavelength? Why do they produce different notes?
• How did old-time stereo components compute the Fourier coefficients for the display?
• How does digital music (mp3s, etc.) use Fourier analysis?
List of useful websites:

* Illuminations page on varying sinusoidal waves: http://illuminations.nctm.org/Activity.aspx?id=3589
* Illuminations page on sound waves: http://illuminations.nctm.org/Activity.aspx?id=4202
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- Illuminations page on sound waves: [http://illuminations.nctm.org/Activity.aspx?id=4202](http://illuminations.nctm.org/Activity.aspx?id=4202)
- Desmos graphing calculator: [https://www.desmos.com/calculator](https://www.desmos.com/calculator)
• Wikipedia page on Fourier analysis: http://en.wikipedia.org/wiki/Fourier_analysis
• MathWorld page on Fourier series: http://mathworld.wolfram.com/FourierSeries.html
• Science Illustrations page on sound waves: http://www.iknowthat.com/ScienceIllustrations/sound/science_desk.swf
• The Physics Hypertextbook page on Music & Noise: http://physics.info/music/
Scott Roberts’ page on Combining Sine Waves to Produce Musical Tones and the Human Voice: http://www.mindspring.com/~scottr/zmusic/

ThinkQuest page, The Soundry Interactive Sound Lab: http://library.thinkquest.org/19537/java/Wave.html

Kyle Forinash’s page of wave and sound links: http://homepages.ius.edu/kforinas/K/Sound/Slinks.html


• HyperPhysics section on Musical Instruments: http://hyperphysics.phy-astr.gsu.edu/hbase/music/musinscon.html