Calculus 2 TAYLOR AND MACLAURIN SERIES

Given a function \( f(x) \) and a point \( x = a \), we wish to approximate \( f(x) \) in the neighborhood of \( x = a \) by a polynomial of degree \( n \).

\[ P_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n = \sum_{k=0}^{n} c_k (x-a)^k \]

We have \( n+1 \) coefficients to choose. We require that \( P_n(x) \) match the function \( f(x) \) and its first \( n \) derivatives at \( x = a \).

\[ f(a) = P_n(a), \quad f'(a) = P'_n(a), \quad f''(a) = P''_n(a), \quad \ldots \quad f^{(n)}(a) = P^{(n)}_n(a) \]

\[ f(a) = P_n(a) = c_0 \quad (=c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n \text{ at } x = a) \]

\[ f'(a) = P'_n(a) = c_1 \quad (=c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + nc_n(x-a)^{n-1} \text{ at } x = a) \]

\[ f''(a) = P''_n(a) = 2c_2 \quad (=2c_2 + 3 \cdot 2c_3(x-a) + 4 \cdot 3c_4(x-a)^2 + \cdots n(n-1)c_n(x-a)^{n-2} \text{ at } x = a) \]

\[ f''(a) = P'''_n(a) = 3 \cdot 2c_3 \quad (=3 \cdot 2c_3 + 4 \cdot 3 \cdot 2c_4(x-a) \cdots n(n-1)(n-2)c_n(x-a)^{n-3} \text{ at } x = a) \]

\[ \vdots \]

\[ f^{(n)}(a) = P^{(n)}_n(a) = n!c_n \quad (=n(n-1)(n-2)(n-3)\ldots1c_n) \]

We solve for the coefficients \( c_0, c_1, \ldots c_n \)

\[ c_0 = f(a), \quad c_1 = f'(a), \quad c_2 = \frac{f''(a)}{2!}, \quad c_3 = \frac{f'''(a)}{3!}, \quad \ldots \quad c_n = \frac{f^{(n)}(a)}{n!} \]

If we define \( f^{(0)} = f \) then the formula \( c_n = \frac{f^{(n)}(a)}{n!} \) holds for all \( n \).

The polynomial \( P_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n = \sum_{k=0}^{n} c_k (x-a)^k \) is called the \( n^{th} \) Taylor polynomial for \( f(x) \) about \( x = a \). Note that \( n \) is the order of the derivative not the number of terms. If the \( n^{th} \) order derivative of \( f(x) \) is zero then the \( n^{th} \) Taylor polynomial, \( P_n(x) \), is not necessarily of degree \( n \). That is why it is commonly called the \( n^{th} \) order Taylor polynomial. For instance the first two Taylor polynomials of \( \cos x \) about \( x = 0 \) are \( P_0(x) = 1 \) and \( P_1(x) = 1 + 0x = 1 \). If \( a = 0 \) the polynomial is called the Maclaurin polynomial.

As \( n \) increases we match higher order derivatives of \( f(x) \) with \( P_n(x) \) at \( x = a \). We might expect that the approximation \( P_n(x) \) of \( f(x) \) improves as \( n \) increases but that is not always the case. We hope that \( \lim_{n \to \infty} P_n(x) = f(x) \). The \( \lim_{n \to \infty} P_n(x) \) is defined to be the Taylor Series of \( f(x) \) about \( x = a \)

The Taylor Series of \( f(x) \) about \( x = a \) is \( \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \). If \( a = 0 \) it is called a Maclaurin Series.
The Taylor Series is a power series about \( x = a \). Therefore there are only three possibilities for the \( x \) interval of convergence of the series.

1. The series converges absolutely for all \( x \). The radius of convergence is \( \infty \).
2. The series converges absolutely only for \( x = a \). The radius of convergence is zero.
3. The series converges absolutely in an interval about \( x = a, (a - R, a + R) \) and diverges outside the interval. The endpoints need to be tested separately. \( R \) is called the radius of convergence. \( 0 < R < \infty \)

For any given Taylor Series for \( f(x) \) we need to determine the following:
1. What is the interval of convergence for \( x \)? We use the ratio test.
2. Does the series converge to \( f(x) \)?

Example: \( f(x) = e^x \). Find the 3rd MacLaurin polynomial, \( P_3(x) \). This is a polynomial expression of degree 3 for \( e^x \) about \( x = 0 \) which matches the value of \( e^x \) and the value of the first and second and third derivatives of \( e^x \) at \( x = 0 \) with the values of the polynomial at \( x = 0 \).

\[
P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3
\]

\[
f(0) = e^0 = 1, \quad f'(0) = e^x \bigg|_{x=0} = 1, \quad f''(0) = e^x \bigg|_{x=0} = 1, \quad f'''(0) = e^x \bigg|_{x=0} = 1
\]

\[
P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}
\]

Find the MacLaurin series of \( e^x \)

\[
f(0) = f'(0) = f''(0) = \cdots f^{(n)}(0) = 1. \text{ The MacLaurin series of } e^x \text{ is } 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.
\]

Does the series converge? \( \lim_{n \to \infty} \left| \frac{x^{n+1}n!}{(n+1)!x^n} \right| = \lim_{n \to \infty} \left| \frac{x}{n+1} \right| = 0 < 1 \). The series converges absolutely for all \( x \) by the ratio test. The question remains, does the Taylor series converge to \( f(x) \)?

Example: Find the Taylor series for \( \sin x \) about \( x = 0 \)
The Taylor series for \( f(x) \) about \( x = 0 \) is
\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n
\]

\[
f(0) = \sin x_{x=0} = 0
\]
\[
f'(0) = \cos x_{x=0} = 1
\]
\[
f''(0) = -\sin x_{x=0} = 0
\]
\[
f'''(0) = -\cos x_{x=0} = -1
\]
\[
f^{(4)}(0) = \sin x_{x=0} = 0
\]
\[
f^{(5)}(0) = \cos x_{x=0} = 1
\]

The Taylor series for \( \sin x \) about \( x = 0 \) is
\[
0 + 1x + \frac{0x^2}{2!} - \frac{1x^3}{3!} + \frac{0x^4}{4!} + \frac{1x^5}{5!} + \frac{0x^6}{6!} - \frac{1x^7}{7!} + \cdots =
\]
\[
x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}
\]
Verify this series converges absolutely for all \( x \).

Example: Find the MacLaurin series of \( \cos x \)

The Taylor series for \( f(x) \) about \( x = 0 \) is
\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n
\]

\[
f(0) = \cos x_{x=0} = 1
\]
\[
f'(0) = -\sin x_{x=0} = 0
\]
\[
f''(0) = -\cos x_{x=0} = -1
\]
\[
f'''(0) = \sin x_{x=0} = 0
\]
\[
f^{(4)}(0) = \cos x_{x=0} = 1
\]
\[
f^{(5)}(0) = -\sin x_{x=0} = 0
\]

The Taylor series for \( \cos x \) about \( x = 0 \) is
\[
1 + 0x - \frac{1x^2}{2!} + \frac{0x^3}{3!} + \frac{1x^4}{4!} + \frac{0x^5}{5!} - \frac{1x^6}{6!} + \frac{0x^7}{7!} + \cdots =
\]
\[
1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}
\]
Verify this series converges absolutely for all \( x \).

Example: Find the Taylor series of \( \sin 2x \) about \( x = 0 \)

The Taylor series for \( f(x) \) about \( x = 0 \) is
\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n
\]

\[
f(0) = \sin 2x_{x=0} = 0
\]
\[
f'(0) = 2 \cos 2x_{x=0} = 2
\]
\[
f''(0) = -2^2 \sin 2x_{x=0} = 0
\]
\[
f'''(0) = -2^3 \cos 2x_{x=0} = -2^3
\]
\[
f^{(4)}(0) = 2^4 \sin 2x_{x=0} = 0
\]
\[
f^{(5)}(0) = 2^5 \cos 2x_{x=0} = 2^5
\]
In order to recognize powers and factorials for formulas it is useful not to multiply out coefficients.

The Taylor series for \( x^2 \sin \) about 0 is

\[
\begin{align*}
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
\end{align*}
\]

Example: Find the Taylor series for \( \ln x \) about \( x = 1 \).

The Taylor Series of \( f(x) \) about \( x = 1 \) is

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x-1)^n
\]

Keep going until you recognize a pattern and can generate a formula for the \( n \)th derivative of \( f \) with respect to \( x \) evaluated at \( x = 1 \), \( f^{(n)}(1) = (-1)^{n+1} (n-1)! \) \( n \geq 1 \)

The Taylor series for \( \ln x \) about \( x = 1 \) is

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1} (n-1)!}{n!} (x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x-1)^n}{n}.
\]

This series converges absolutely for \( |x-1| < 1 \), diverges at \( x = 0 \) and converges conditionally at \( x = 2 \).

Example: Find the Taylor series of \( f(x) = x^3 - 3x^2 + 2x + 1 \) about \( x = 0 \)

\[
\begin{align*}
f(0) &= 1 \\
f'(x) &= 3x^2 - 6x + 2 \quad x=0 = 2 \\
f'(0) &= 6x - 6 \quad x=0 = -6 \\
f'(0) &= 6 \\
f^{(n)}(0) &= 0 \quad n \geq 4
\end{align*}
\]

The Taylor series of \( f(x) \) is

\[
1 + 2x - \frac{6x^2}{2!} - \frac{6x^3}{3!} = 1 + 2x - 3x^2 + x^3
\]

of course!!

If a power series of \( x \) converges to \( f(x) \) on some open interval about \( x = 0 \) then the power series is the Taylor series.

Quiz: Find the Taylor series of \( \ln(1 + x) \) about \( x = 0 \).
You will need to memorize the following Taylor series about $x = 0$ (MacLaurin series). The interval of convergence is also given.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (-\infty, \infty)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad (-\infty, \infty)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad (-\infty, \infty)$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \quad (-1, 1]$$

$$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + x^4 - \cdots = \sum_{n=0}^{\infty} (-1)^n x^n \quad (-1, 1)$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)} \quad [-1, 1]$$

For homework problems see [http://voyager.dvc.edu/~lmonth/Calc2/Hw4Series.pdf](http://voyager.dvc.edu/~lmonth/Calc2/Hw4Series.pdf)