STRATEGY FOR DETERMINING THE CONVERGENCE OF POSITIVE SERIES

1. If it looks like the series terms do not converge to zero use the Divergence Test. (Only use if you can quickly check if \( \lim_{n \to \infty} a_n \neq 0 \))

2. \( p \)-series (\( \sum \frac{1}{n^p} \)) or geometric series (\( \sum_{n=1}^{\infty} ar^n \))
P series will converge only if \( p > 1 \). Geometric series will converge only if \( |r| < 1 \).
Use algebra to get geometric series in right form.

3. If the series is similar to p-series or geometric series try the Comparison Test.
POSITIVE SERIES ONLY

4a. If the inequality goes the wrong way in the Comparison Test or the series is a ratio of polynomials or algebraic functions try the Limit Comparison Test/Extended Limit Comparison Test. Note: This is not the only case where limit comparison is useful. \( \sum \sin \left( \frac{1}{n} \right) \) diverges by limit comparison with the divergent harmonic series \( \sum \frac{1}{n} \).
POSITIVE SERIES ONLY

4b. If \( a_n = 0 \) or \( a_n = \infty \) in the limit comparison test try the Extended Limit Comparison Test. POSITIVE SERIES ONLY

5. If the series contains factorials or exponentials. Try the Ratio Test.

6. If \( a_n = f(n) \) and \( f(x) \) is a positive decreasing function and \( \int_1^\infty f(x)dx \) can be evaluated try the Integral Test.
POSITIVE SERIES ONLY
Calculus 2  CONVERGENCE TESTS FOR SERIES OF POSITIVE TERMS

0. Divergence Test
    if \( \lim_{n \to \infty} a_n \neq 0 \) then \( \sum a_n \) diverges

1. Integral Test for \( \sum a_n \)
   Let \( f(n) = a_n \), where \( f(x) \) is continuous, positive, decreasing on \([1, \infty)\)
   if the improper integral \( \int_1^\infty f(x) \, dx \) converges then \( \sum a_n \) converges
   if the improper integral \( \int_1^\infty f(x) \, dx \) diverges then \( \sum a_n \) diverges

2. Comparison Test for series of positive terms (eventually)
   if \( 0 < a_n \leq b_n \) for all sufficiently large \( n \) and \( \sum b_n \) converges then \( \sum a_n \) converges
   if \( a_n \geq b_n > 0 \) for all sufficiently large \( n \) and \( \sum b_n \) diverges then \( \sum a_n \) diverges

3. Limit Comparison Test / Extended Limit Comparison Test for series of positive terms (eventually)
   1. if \( \lim_{n \to \infty} \frac{a_n}{b_n} = c \), and \( 0 < c < \infty \) then \( \sum a_n \) and \( \sum b_n \) both converge or both diverge.
   2. if \( \lim_{n \to \infty} \frac{a_n}{b_n} = 0 \) and \( \sum b_n \) converges, then \( \sum a_n \) converges
   3. if \( \lim_{n \to \infty} \frac{a_n}{b_n} = \infty \) and \( \sum b_n \) diverges, then \( \sum a_n \) diverges.

4. Ratio Test for \( \sum a_n \) (Useful for factorial and exponential)
   Let \( r = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \).
   if \( r < 1 \) then \( \sum |a_n| \) converges \( \sum a_n \) converges
   if \( r > 1 \) or \( r \) is infinite then \( \sum a_n \) diverges
   if \( r = 1 \) then no information about the convergence of \( \sum a_n \) results.
Examples using the ratio, limit comparison tests, and extended limit comparison tests.

1. \(\sum \frac{1}{n!}\) converges by the ratio test
   \[
   \lim_{n \to \infty} \frac{n!}{(n+1)!} = \lim_{n \to \infty} \frac{1}{n+1} = 0 < 1
   \]

2. \(\sum \frac{2^n}{n!}\) converges by the ratio test
   \[
   \lim_{n \to \infty} \frac{n+1}{n+2} = \lim_{n \to \infty} \frac{2^n}{n+1} = 0 < 1
   \]

3. \(\sum \frac{n!}{(2n)!}\) converges by the ratio test. Be careful \((2n)! \neq 2 \cdot n!\)
   \[
   \lim_{n \to \infty} \frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!} = \lim_{n \to \infty} \frac{n+1}{2n+2} = 0 < 1
   \]

4. The ratio test is inconclusive for \(\sum \frac{1}{n^2 + 3n}\)
   \[
   \lim_{n \to \infty} \frac{n^2 + 3n}{n^2 + (n+1)^2 + 3(n+1)} = \lim_{n \to \infty} \frac{n^2 + 3n}{n^2 + 5n + 4} = \lim_{n \to \infty} \frac{1 + \frac{1}{n}}{1 + \frac{5}{n} + \frac{4}{n^2}} = 1
   \]
   The ratio test fails if \(a_n\) is a rational or algebraic function of \(n\) because \(\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1\)

TRY THE LIMIT COMPARISON TEST FOR SERIES WITH POLYNOMIAL OR ALGEBRAIC EXPRESSIONS.
Find the ratio of the highest power term in numerator to highest power term in the denominator and use this as the nth term of the comparison series.

ALSO TRY THE LIMIT COMPARISON TEST IF THE INEQUALITY GOES THE WRONG WAY IN THE DIRECT COMPARISON TEST

5. Compare \(\sum a_n = \sum \frac{1}{n^2 - 3n}\) to the convergent series \(\sum b_n = \sum \frac{1}{n^2}\)
   \[
   \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^2}{n^2 - 3n} = 1 < \infty \quad \text{and} \quad \sum \frac{1}{n^2} \text{ converges}
   \]
   \[
   \sum \frac{1}{n^2 - 3n} \text{ converges by the limit comparison test}
   \]

6. Compare \(\sum a_n = \sum_{n=1}^{\infty} \frac{1}{n + \ln n}\) to the divergent series \(\sum b_n = \sum_{n=1}^{\infty} \frac{1}{n}\)
   \[
   \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n}{n + \ln n} = \lim_{n \to \infty} \frac{1}{1 + \frac{\ln n}{n}} = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}
   \]
   \[
   \sum_{n=1}^{\infty} \frac{1}{n + \ln n} \text{ diverges by the limit comparison test.}
   \]

7. Compare \(\sum a_n = \sum \frac{1}{\sqrt[3]{n^2 + 5n + 7}}\) to the divergent series \(\sum b_n = \sum \frac{1}{n^{2/3}}\)
   \[
   \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^{2/3}}{\sqrt[3]{n^2 + 5n + 7}} = 1 < \infty \quad \text{and} \quad \sum b_n = \sum \frac{1}{n^{2/3}} \text{ diverges}
   \]
   \[
   \sum \frac{1}{n^{2/3}} \text{ diverges by the limit comparison test}
   \]
8. Compare \( \sum a_n = \sum \frac{5n^5 + 3n^2 + 3n - 7}{6n^7 + 2n^5 + 9n} \) to the convergent series \( \sum b_n = \sum \frac{1}{n^2} \)

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{(5n^5 + 3n^2 + 3n - 7)n^2}{6n^7 + 2n^5 + 9n} = \frac{5}{6} < \infty \quad \text{and} \quad \sum \frac{1}{n^2} \text{ converges}
\]

\[
\sum \frac{5n^5 + 3n^2 + 3n - 7}{6n^7 + 2n^5 + 9n} \text{ converges by the limit comparison test}
\]

9. Compare \( \sum a_n = \sum_{n=1}^{\infty} \frac{2 + \sqrt[3]{32n^6 + 1}}{n^{2/3} n^3 + n^2} \) to the divergent series \( \sum b_n = \sum_{n=1}^{\infty} \frac{1}{n^{2/15}} \)

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{2 + \sqrt[3]{32n^6 + 1}}{n^{2/3} n^3 + n^2} \frac{n^{4/3}}{n^{4/3}} = \frac{2}{3} \quad \text{and} \quad \sum \frac{1}{n^{2/15}} \text{ diverges}.
\]

\[
\sum_{n=1}^{\infty} \frac{2 + \sqrt[3]{32n^6 + 1}}{n^{2/3} n^3 + n^2} \text{ diverges by the limit comparison test.}
\]

10. Compare \( \sum a_n = \sum_{n=1}^{\infty} \frac{(\ln n)^3}{n^2} \) to the convergent series \( \sum b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \)

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{(\ln n)^3}{n^{3/2}} = 0 \quad \text{and} \quad \sum \frac{1}{n^{3/2}} \text{ converges}
\]

\[
\sum_{n=1}^{\infty} \frac{(\ln n)^3}{n^2} \text{ converges by the extended limit comparison test.}
\]

11. Compare \( \sum a_n = \sum_{n=1}^{\infty} \frac{\ln n}{n^3} \) to the convergent series \( \sum b_n = \sum_{n=1}^{\infty} \frac{1}{n^2} \)

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\ln n}{n^3} = 0 \quad \text{and} \quad \sum \frac{1}{n^2} \text{ converges}
\]

\[
\sum_{n=1}^{\infty} \frac{\ln n}{n^3} \text{ converges by the extended limit comparison test.}
\]

12. Compare \( \sum a_n = \sum_{n=1}^{\infty} \frac{\ln n}{\sqrt[n]{e^n}} \) to the convergent series \( \sum b_n = \sum_{n=1}^{\infty} \frac{1}{e^n} \)

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\ln n}{\sqrt[n]{e^n}} = 0 \quad \text{and} \quad \sum \frac{1}{e^n} \text{ converges}
\]

\[
\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt[n]{e^n}} \text{ converges by the extended limit comparison test.}
\]

13. Compare \( \sum a_n = \sum_{n=1}^{\infty} \frac{(\ln n)^2}{\sqrt[n]{n}} \) to the divergent series \( \sum b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}} \)

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{(\ln n)^2}{\sqrt[n]{n}} = \infty \quad \text{and} \quad \sum \frac{1}{\sqrt[n]{n}} \text{ diverges}
\]

\[
\sum_{n=1}^{\infty} \frac{(\ln n)^2}{\sqrt[n]{n}} \text{ diverges by the extended limit comparison test.}
\]

14. Compare \( \sum a_n = \sum_{n=1}^{\infty} \frac{1}{\ln n} \) to the divergent series \( \sum b_n = \sum_{n=1}^{\infty} \frac{1}{n} \)

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{1}{\ln n} = \infty \quad \text{and} \quad \sum \frac{1}{n} \text{ diverges}
\]

\[
\sum_{n=1}^{\infty} \frac{1}{\ln n} \text{ diverges by the extended limit comparison test.}
\]

15. Compare \( \sum a_n = \sum_{n=1}^{\infty} \frac{\ln n}{n} \) to the divergent series \( \sum b_n = \sum_{n=1}^{\infty} \frac{1}{n} \)

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\ln n}{n} = \infty \quad \text{and} \quad \sum \frac{1}{n} \text{ diverges}
\]

\[
\sum_{n=1}^{\infty} \frac{\ln n}{n} \text{ diverges by the extended limit comparison test.}
\]

For homework problems see
http://voyager.dvc.edu/~LMonth/Calc2/Hw2aSeries.pdf
http://voyager.dvc.edu/~LMonth/Calc2/HW2bSeries.pdf