Calculus 2  MONOTONE SEQUENCES

If \( a_{n+1} \geq a_n \) for all \( n \in \mathbb{N} \) we say \( \{a_n\} \) is increasing. If \( a_{n+1} > a_n \) for all \( n \in \mathbb{N} \) we say \( \{a_n\} \) is strictly increasing. Similarly for decreasing and strictly decreasing. Some people in order to avoid ambiguity use terms nondecreasing for the former and increasing for the latter. A sequence is said to be monotonic if it is either nondecreasing or nonincreasing and strictly monotonic if in addition consecutive terms are never equal. If the inequalities hold for all \( n \geq n_0 \) we say the sequence \( \{a_n\} \) is eventually monotonic. Recall that dropping a finite number of terms from a sequence does not affect its convergence property (whether it converges or diverges). The convergence property of \( \{a_n\}_{n_0}^\infty \) is the same as the convergence property of \( \{a_n\}_n^\infty \).

**MONOTONE CONVERGENCE THEOREM (MCT)**

If a sequence \( \{a_n\} \) is monotone nondecreasing it satisfies one of the convergence properties below.
1. It increases without bound and is therefore divergent. The \( \lim_{n \to \infty} a_n = \infty \).
2. It is bounded above by \( M \) \( (a_n \leq M \) for all \( n \in \mathbb{N} \)) and therefore \( \lim_{n \to \infty} a_n = L \leq M \). The sequence converges to a number less than or equal to \( M \).

Eventual monotonicity is all that we really need for MCT because the convergence property of \( \{a_n\}_1^\infty \) is the same as the convergence property of \( \{a_n\}_{n_0}^\infty \).

A direct consequence of MCT is that any decreasing sequence of positive terms is convergent (or any nonincreasing sequence of nonnegative terms) because positive terms are bounded below by zero.

**IF A SEQUENCE IS POSITIVE AND CONVERGES TO ZERO THAT DOES NOT MEAN IT IS DECREASING**

Example: Constant sign and convergence to zero do not imply monotonicity.
\( a_n = |\sin n|/n \) is positive and \( \lim_{n \to \infty} a_n = 0 \) but the terms are not eventually decreasing they are oscillating.

We prove a sequence is increasing by proving that \( a_{n+1} > a_n \)
\[ \text{difference} \quad a_{n+1} - a_n > 0. \]
\[ \text{ratio} \quad a_{n+1}/a_n > 1 \] (only for sequence of positive terms)
\[ \text{derivative} \quad \text{Let } f(n) = a_n, f'(x) > 0 \] (if the derivative exists)

Similarly for decreasing.
Example: \(\{a_n\} = \left\{ \frac{n}{n+1} \right\}\) Is the sequence increasing or decreasing eventually?

\[
a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{1}{(n+1)(n+2)} > 0. \text{ The sequence is increasing.}
\]

\[
a_{n+1} = (n+1)(n+1) > 1. \text{ The sequence is increasing.}
\]

\[
f(x) = \frac{x}{x+1}, \quad f'(x) = \frac{1}{(x+1)^2} > 0. \text{ The sequence is increasing.}
\]

It is easy to see in this case

\[
\lim_{n \to \infty} \frac{n}{n+1} = 1
\]

and

\[
\frac{n}{n+1} < 1
\]

The terms are increasing to 1 from below.

Be careful.

**GIVEN A SEQUENCE OF POSITIVE TERMS** \(a_n = \frac{f(n)}{g(n)}\) WHERE \(f(n)\) AND \(g(n)\) ARE POSITIVE.

\(f(n) < g(n)\) DOES NOT IMPLY THE SEQUENCE IS DECREASING. IT MERELY IMPLIES THE TERMS ARE LESS THAN 1

Example: \(\{a_n\} = \left\{ \frac{n}{n^2+1} \right\}\)

\[
f(x) = \frac{x}{x^2+1}
\]

\[
f'(x) = \frac{x^2+1-x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0. \text{ The sequence } \left\{ \frac{n}{n^2+1} \right\} \text{ is decreasing.}
\]

The sequence \(\left\{ \frac{n}{n^2+1} \right\}\) is decreasing and bounded below by zero. \(\left\{ \frac{n}{n^2+1} \right\}\) converges by MCT. In this case it is easy to see

\[
\lim_{n \to \infty} \frac{n}{n^2+1} = 0.
\]

Example: \(\{a_n\} = \left\{ \frac{n^2}{e^n} \right\}\)

\[
f(x) = \frac{x^2}{e^x}
\]

\[
f'(x) = \frac{2xe^x-x^2e^x}{e^{2x}} = \frac{xe^{x(2-x)}}{e^{2x}} < 0 \text{ if } x > 2. \text{ The sequence } \left\{ \frac{n^2}{e^n} \right\} \text{ is decreasing eventually.}
\]

The sequence \(\left\{ \frac{n^2}{e^n} \right\}\) is decreasing and bounded below by zero. \(\left\{ \frac{n^2}{e^n} \right\}\) converges by MCT. In this case it is easy to see

\[
\lim_{n \to \infty} \frac{n^2}{e^n} = 0.
\]

Example: \(\{a_n\} = \left\{ \frac{2n}{3n+1} \right\}\)

\[
a_{n+1} = \frac{(2n+2)(3n+1)}{2n(3n+4)} = \frac{6n^2+8n+2}{6n^2+8n} > 1. \text{ The sequence } \{a_n\} \text{ is increasing.}
\]

Also

\[
a_n < 1. \text{ The sequence } \{a_n\} \text{ is bounded above by 1.}
\]

The monotone convergence theorem tells us the sequence converges to a limit \(\leq 1\) because the sequence \(\{a_n\}\) is increasing and bounded above by 1. In this case we can easily find the limit of the sequence.

\[
\lim_{n \to \infty} \frac{2n}{3n+1} = \frac{2}{3}
\]
Example: \( \{a_n\} = \left\{ \frac{2n+1}{3n+1} \right\} \)

\[
\frac{a_{n+1}}{a_n} = \frac{(2n+3)(3n+1)}{(2n+1)(3n+4)} = \frac{6n^2+11n+3}{6n^2+11n+4} < 1.
\]
The sequence \( \{a_n\} \) is decreasing and bounded below by zero.

The sequence converges by MCT. In this case we can easily find the limit of the sequence.

\[
\lim_{n \to \infty} \frac{2n}{3n+1} = \frac{2}{3}
\]

Let's look at the list of the previous examples with labels \( I \) for increasing and \( D \) for decreasing.

\[
\left\{ \frac{n}{n+1} \right\} \text{ is } I, \quad \left\{ \frac{n}{n^2+1} \right\} \text{ is } D, \quad \left\{ \frac{n^2}{e^n} \right\} \text{ is } I, \quad \left\{ \frac{2n+1}{3n+1} \right\} \text{ is } D
\]

We already noted that given a sequence of positive terms \( a_n = \frac{f(n)}{g(n)} \) where \( f(n) \) and \( g(n) \) are positive.

\( f(n) < g(n) \) does not imply the sequence is decreasing. It merely implies the terms are less than 1. You might think that if we also have \( f'(n) < g'(n) \) then the sequence is decreasing. But looking at the above examples you see that this is not the case. In fact you might find it surprising that \( \left\{ \frac{2n}{3n+1} \right\} \) and \( \left\{ \frac{2n+1}{3n+1} \right\} \) have different monotone properties. Are there sufficient condition for monotonicity?

Consider a sequence \( a_n = \frac{f(n)}{g(n)} \) where \( f(x) \) and \( g(x) \) are positive differentiable functions defined for all \( x \geq n_0 \) and the values of \( f(x) \) and \( g(x) \) agree with the values of \( f(n) \) and \( g(n) \), for \( n \geq n_0 \). The sequence \( \{a_n\} \) is decreasing if and only if \( \frac{f'}{g'} < 0 \).

\[
\begin{align*}
\left( \frac{f}{g} \right)' &= \frac{f'g - fg'}{g^2} < 0 \\
\text{sgn} \left( \frac{f}{g} \right)' &= \text{sgn} (f'g - fg') < 0 \\
f'g < fg' \\
f' < g' \\
\frac{f}{g} < \frac{g}{f} \\
\text{(ln f)'} < (\text{ln g})'
\end{align*}
\]

**Theorem:** Given a sequence \( a_n = \frac{f(n)}{g(n)} \) where \( f(x) \) and \( g(x) \) are positive differentiable functions for all \( x \geq n_0 \) and \( f(x) \) agrees with \( a_n \) for \( n \geq n_0 \). The sequence \( \{a_n\} \) is decreasing if and only if \( (\text{ln f})' < (\text{ln g})' \) (increasing iff \( (\text{ln f})' > (\text{ln g})' \)).

Re-examining the two examples \( \frac{2n}{3n+1} \) and \( \frac{2n+1}{3n+1} \) we see by factoring that although both sequences converge to 2/3 the first is always less than 2/3 and the second is always greater than 2/3. So maybe the results are not so surprising after all!!

\[
\frac{2n}{3n+1} = \frac{2}{3} \frac{n}{n+1/3} < 2/3 \\
\text{and} \\
\frac{2n+1}{3n+1} = \frac{2}{3} \frac{n+1/2}{n+1/3} > 2/3.
\]

Example: \( a_n = \frac{2^n}{n!} \)

We can’t let \( f(x) = \frac{2^x}{x!} \) because \( x! \) doesn’t exist!!

\[
\frac{a_{n+1}}{a_n} = \frac{2^{n+1}n!}{(n+1)!2^n} = \frac{2^{n+1}}{n+1} \cdot \frac{n!}{n!} < 2 \text{ for } n > 1.
\]

The sequence is nonincreasing for \( n \geq 1 \).
(The 1st and 2nd terms are equal.) The sequence \( \left\{ \frac{2^n}{n!} \right\} \) is nonincreasing. The sequence \( \left\{ \frac{2^n}{n!} \right\} \) is also bounded below by zero because all the terms are positive. By the monotone convergence theorem \( \left\{ \frac{2^n}{n!} \right\} \) converges (to a limit \( \geq 0 \)).

Example: \( \{a_n\} = \left\{ \frac{10^n}{n!} \right\} \)

\[
\frac{a_{n+1}}{a_n} = \frac{10^{n+1}n!}{(n+1)!10^n} = \frac{10^n \cdot 10}{n+1} = \frac{10}{n+1} < 1 \quad \text{for } n > 9. \quad \text{The sequence is nonincreasing for } n \geq 9.
\]

(The 9th and 10th terms are equal.) The sequence \( \left\{ \frac{10^n}{n!} \right\} \) is nonincreasing. The sequence \( \left\{ \frac{10^n}{n!} \right\} \) is also bounded below by zero because all the terms are positive. By the monotone convergence theorem \( \left\{ \frac{10^n}{n!} \right\} \) converges (to a limit \( \geq 0 \)).

Proof: \( \lim_{n \to \infty} \frac{10^n}{n!} = 0 \)

Let

\[
L = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} a_n \quad \text{The monotone convergence theorem tells us } L \text{ exists.}
\]

\[
a_{n+1} = \frac{10}{n+1} a_n \quad \text{from our previous calculation}
\]

\[
L = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \frac{10}{n+1} a_n = \lim_{n \to \infty} \frac{10}{n+1} \cdot \lim_{n \to \infty} a_n
\]

\[
L = \lim_{n \to \infty} \frac{10}{n+1} \cdot L = 0 \cdot L = 0
\]

\( \lim_{n \to \infty} \frac{10^n}{n!} = 0 \) means \( n! \) grows faster than \( 10^n \) (or any base to the \( n \) for that matter).

Example: \( \lim_{n \to \infty} \frac{x^n}{n!} = 0 \) (for any real fixed \( x \) )

Proof:

1. If \( x = 0 \) then \( \lim_{n \to \infty} \frac{x^n}{n!} = 0. \)

2. \( \lim_{n \to \infty} \frac{|x|^n}{n!} = 0, \quad x \neq 0. \)

Proof:

Let

\[
a_n = \frac{|x|^n}{n!}
\]

\[
a_{n+1} = \frac{|x|^{n+1}}{(n+1)!} = \frac{|x|^n \cdot |x|}{(n+1)!} = \frac{|x|^n}{n+1} < 1 \quad \text{for } n > |x| - 1.
\]

The sequence \( \left\{ \frac{|x|^n}{n!} \right\} \) is decreasing and bounded below by zero.

By the monotone convergence theorem \( \lim_{n \to \infty} \frac{|x|^n}{n!} = L \geq 0. \)

Let

\[
L = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} a_n \quad \text{The monotone convergence theorem tells us } L \text{ exists.}
\]

\[
a_{n+1} = \frac{|x|^n}{n+1} a_n \quad \text{from our previous calculation}
\]

\[
L = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \frac{|x|^n}{n+1} \cdot \lim_{n \to \infty} a_n
\]

\[
L = \lim_{n \to \infty} \frac{|x|^n}{n+1} \cdot L = 0 \cdot L = 0
\]

3. \( \lim_{n \to \infty} \frac{x^n}{n!} = 0 \)

Proof:

\[
\frac{-|x|^n}{n!} \leq \frac{x^n}{n!} \leq \frac{|x|^n}{n!}
\]

\[
\lim_{n \to \infty} \frac{-|x|^n}{n!} = \lim_{n \to \infty} \frac{|x|^n}{n!} = 0
\]

\[
\lim_{n \to \infty} \frac{x^n}{n!} = 0 \quad \text{Squeeze theorem}
\]
\[
\lim_{n \to \infty} \frac{x^n}{n!} = 0 \text{ means } n! \text{ grows faster than } x^n \text{ for any real fixed } x.
\]

**FACTOTIAL TRUMPS EXPONENTIAL**

Example: What about \( \lim_{n \to \infty} \frac{n^n}{n!} \)?

It is easier to consider \( a_n = \frac{n!}{n^n} \)

\[
a_n = \frac{n!}{n^n} = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \cdots \cdot \frac{1}{n} < \frac{1}{n}
\]

and

\[
\lim_{n \to \infty} \frac{1}{n} = 0
\]

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n!}{n^n} = 0.
\]

and all the terms are positive, therefore

\[
\lim_{n \to \infty} \frac{n^n}{n!} = \infty
\]

We can also show the sequence \( \left\{ \frac{n!}{n^n} \right\} \) is monotone decreasing.

\[
a_{n+1} = \frac{(n+1)n^n}{(n+1)n^{n+1}} = \frac{n^n}{(n+1)n^n} = \frac{n^n}{n^{n+1}} = \left( \frac{n}{n+1} \right)^n < 1. \quad \left( \left| \frac{n}{n+1} \right| < 1 \right)
\]

The sequence \( \{a_n\} = \left\{ \frac{n!}{n^n} \right\} \) is decreasing and bounded below by zero. By MCT \( \frac{n!}{n^n} \) converges (to a limit \( \geq 0 \)).

Example: Another way to prove \( \lim_{n \to \infty} \frac{n!}{n^n} = 0 \)

\[
L = \lim_{n \to \infty} a_n = \lim_{n \to \infty} a_{n+1} \geq 0. \quad \text{The monotone convergence theorem tells us } L \text{ exists.}
\]

\[
a_{n+1} = \left( \frac{n+1}{n} \right)^n a_n. \quad \text{from our previous calculation}
\]

Take \( \lim_{n \to \infty} \) of both sides.

\[
L = \frac{1}{e} \cdot L
\]

which has solution

\[
L = 0.
\]

*(We have used the result that \( \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = e \))*

To evaluate

\[
\lim_{n \to \infty} \left( \frac{n}{n+1} \right)^n = \lim_{n \to \infty} \frac{1}{\left( \frac{n+1}{n} \right)^n} = \frac{1}{\lim_{n \to \infty} \left( \frac{n+1}{n} \right)^n} = \frac{1}{e}
\]

\[
\lim_{n \to \infty} \frac{n!}{n^n} = 0 \text{ means that } n^n \text{ grows faster than } n!.
\]

Be sure to see the handout on factorials and bounds. [http://voyager.dvc.edu/~lmonth/Calc2/HdBounds.pdf](http://voyager.dvc.edu/~lmonth/Calc2/HdBounds.pdf)

For \( n \) large enough

\[
\left( \ln n \right)^R < n^\alpha < x^n < n! < n^n \quad x > 1 \text{ fixed}
\]

The point is that \( R \) can be arbitrarily large and \( \alpha \) can be arbitrarily small positive.

For example

\[
\left( \ln n \right)^{100} < n^{.001}
\]

Eventually

For \( n \) large enough

\[
\left( \ln n \right)^R < n^\alpha < n^{\ln n} < e^n < n! < n^n
\]

The inequality \( n^{\ln n} < e^n \) follows by noting that \( n = e^{\ln n} \) and therefore \( n^{\ln n} = e^{(\ln n)^2} \)
The bounds $([l n n]^R < n^n < x^n < n! < n^n)$ (x > 1) apply to the growth rates as well.

This means in addition to

\[ \ln n^{100} < n^{0.01} \]

it is also true that

\[ \lim_{n \to \infty} \frac{\ln n^{100}}{n^{0.01}} = 0 \]

which is equivalent to the much stronger result that eventually

\[ \ln n^{100} < \varepsilon n^{0.01} \]

for any arbitrarily small positive \( \varepsilon \)

Similarly in addition to

\[ n! > 10^n \]

it is also true that

\[ \lim_{n \to \infty} \frac{n!}{10^n} = \infty \]

which is equivalent to the much stronger result that eventually

\[ n! > M 10^n \]

for any arbitrarily large positive \( M \)

We close with two well known results

\[ n! \sim \frac{n^n}{e^{n-1}} \sqrt{2\pi n} \text{ Stirling's approximation for large } n \]

which means

\[ \lim_{n \to \infty} \frac{n!}{e^{n} \sqrt{2\pi n}} = 1 \]

and

\[ \frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n} \text{ for } n > 1 \]

Proof:

\[ \frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n} \text{ for } n > 1 \]

\[ \ln n! = \ln(2 \cdot 3 \cdot 4 \cdots n) = \ln 2 + \ln 3 + \ln 4 \cdots \ln n \]

We can bound \( \ln n! \) by by bounding the area under the graph of \( \ln x \) above and below by the sum of the areas of rectangles of height \( \ln k \) and width 1. The area of each rectangle is \( \ln k \) and the sum of the areas of the rectangles is \( \ln 2 + \ln 3 + \ln 4 \cdots \ln n = \ln n! \). See the figures.
\[ n^n e^{-n+1} < n! < (n + 1)^{n+1} e^{-n} \]

\[ \frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n} \]

Example: \( \lim_{n \to \infty} \frac{n \sqrt[n]{n!}}{n} = \frac{1}{e} \)

\[ \frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n} \]

\[ \left( \frac{n^n}{e^{n-1}} \right)^{1/n} < \sqrt[n]{n!} < \left( \frac{(n+1)^{n+1}}{e^n} \right)^{1/n} \]

\[ \frac{n}{e^{1-1/n}} < \sqrt[n]{n!} < \frac{(n+1)(n+1)^{1/n}}{en} \]

\[ \lim_{n \to \infty} \frac{1}{e^{1-1/n}} = \lim_{n \to \infty} e^{1-1/n} = \frac{1}{e} \lim_{n \to \infty} 1-1/n = \frac{1}{e} \cdot 1 = \frac{1}{e} \] (\( e^x \) is continuous at \( x = 1 \))

and

\[ \lim_{n \to \infty} \frac{(n+1)(n+1)^{1/n}}{en} = \frac{1}{e} \cdot \lim_{n \to \infty} \frac{n+1}{n} \cdot \lim_{n \to \infty} (n + 1)^{1/n} = \frac{1}{e} \cdot 1 \cdot 1 = \frac{1}{e} \]

\[ \lim_{n \to \infty} \frac{n \sqrt[n]{n!}}{n} = \frac{1}{e} \textit{ Squeeze theorem.} \]

(Use L'Hôpital's rule to evaluate \( \lim_{n \to \infty} (n + 1)^{1/n} \) [http://voyager.dvc.edu/~lmonth/Calc2/HdIndeterm.pdf]

\[ \ln(n + 1)^{1/n} = \frac{\ln(n+1)}{n} \]

\[ \lim_{n \to \infty} \ln(n + 1)^{1/n} = \lim_{n \to \infty} \frac{\ln(n+1)}{n} = 0 \]

\[ \lim_{n \to \infty} (n + 1)^{1/n} = \lim_{n \to \infty} e^{\ln(n+1)^{1/n}} = e^{\lim_{n \to \infty} \ln(n+1)^{1/n}} = e^0 = 1 \quad (e^x \text{ is continuous at } x = 0) \]

Example: The sequence \( \{a_n\} = \left\{ \frac{n^n}{(n/e)^n} \right\} \) is decreasing

\[ \frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}n!e^n}{(n+1)!e(n+1)n^n} = \frac{(n+1)^{n+1}n!e^n}{(n+1)!e(n+1)n^n} \]

\[ = \frac{(n+1)^n}{n!e^n} \cdot \frac{(1+1/n)^n}{e} < 1. \]

(The last inequality follows from the fact that

\[ \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e \]

and

\[ \left\{ \left(1 + \frac{1}{n}\right)^n \right\} \text{ is increasing.} \]

\[ \textit{The proof that } \left\{ \left(1 + \frac{1}{n}\right)^n \right\} \text{ is increasing is a famous proof by Riemann using the binomial theorem.} \]

It follows from MCT that the sequence \( \left\{ \frac{n^n}{n!e^n} \right\} \) converges.

Finding the limit of the sequence follows from Stirling's formula.

\[ \lim_{n \to \infty} \frac{n!}{e^{n\sqrt{2\pi n}}} = 1 \]

\[ \lim_{n \to \infty} \frac{n^n}{n!e^n} = \lim_{n \to \infty} \frac{n^n}{n!\sqrt{2\pi n}} = \lim_{n \to \infty} \frac{n^n\sqrt{2\pi n}}{n!} \cdot \lim_{n \to \infty} \frac{1}{\sqrt{2\pi n}} = 1 \cdot 0 = 0 \]
FYI

\[
\begin{align*}
\lim_{n \to \infty} \frac{\ln n}{n} &= 0 \\
\lim_{n \to \infty} n^{1/n} &= 1 \\
\lim_{n \to \infty} x^{1/n} &= 1 \quad (x > 0) \\
\lim_{n \to \infty} x^n &= 0 \quad |x| < 1 \\
\lim_{n \to \infty} x^n &= \infty \quad (x > 1) \\
\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n &= e^x \quad \text{any } x \\
\lim_{n \to \infty} \frac{x^n}{n!} &= 0 \quad \text{any } x \\
x \text{ is fixed}
\end{align*}
\]

\[
\begin{align*}
\lim_{n \to \infty} \frac{n!}{x^n} &= \infty \quad x \neq 0 \\
\lim_{n \to \infty} \frac{n!}{n^n} &= 0 \\
\lim_{n \to \infty} \frac{n!e^n}{n^n} &= \infty \\
\lim_{n \to \infty} \frac{n!e^n}{n^n \sqrt{n}} &= \sqrt{2\pi} \\
\lim_{n \to \infty} \frac{n!e^n}{n^n \sqrt{2\pi} \sqrt{n}} &= 1
\end{align*}
\]

For homework problems see [http://voyager.dvc.edu/~lmonth/Calc2/Hw2Seq.pdf](http://voyager.dvc.edu/~lmonth/Calc2/Hw2Seq.pdf)