Calculus 2  ESTIMATING WITH TAYLOR SERIES

We now use remainder estimates to approximate function values.

Example: Approximate \( \sin 5 \) to 12 place accuracy.

Consider the Taylor series of \( \sin x \) about \( x = 0 \).

\[
f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)!}.
\]

We want \( |R_n(x)| < 5 \times 10^{-13} \).

\[
|R_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \right| \leq \left| \frac{x^{n+1}}{(n+1)!} \right| < 5 \times 10^{-13}.
\]

At \( x = 5 \) \( |R_n(5)| \leq \frac{5^{n+1}}{(n+1)!} < 5 \times 10^{-13} \). To find \( n \) build a table of \( n \) vs. \( \frac{5^{n+1}}{(n+1)!} \) and find the first time that \( \frac{5^{n+1}}{(n+1)!} < 5 \times 10^{-13} \). You should find that \( n = 31 \). The Taylor series of \( \sin x \), truncated at Taylor polynomial of order 31, evaluated at \( x = 5 \) will approximate \( \sin 5 \) to at least 12 decimal places.

\[
\sin 5 \approx 5 - \frac{5^3}{3!} + \frac{5^5}{5!} - \frac{5^7}{7!} - \cdots - \frac{5^{31}}{31!} = -0.958924274663. \text{ This is the calculator value.}
\]

Example: Estimate \( \pi \) to 2 decimal places.

Consider the Taylor series of \( \tan^{-1} x \) about \( x = 0 \).

\[
f(x) = \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)} \text{ on } [-1, 1].
\]

\( 4 \tan^{-1} 1 = \pi = 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4}{(2n-1)}. \) This is an alternating series. The magnitude of the truncation error is bounded by the magnitude of next term. The \( n^{th} \) term of the series for \( \pi \) has magnitude \( |a_n| = \frac{4}{2n-1} \).

\[
|S - S_n| = |a_{n+1}| = \frac{4}{2n+1} < 5 \times 10^{-3} \text{ (2 decimal place accuracy)}
\]

\[
2n + 1 > \frac{4}{5 \times 10^{-3}}
\]

\[
2n + 1 > 800 \quad 2n > 799 \quad n \geq 400
\]

To get 2 decimal place accuracy for \( \pi \), truncate the Taylor series for \( 4 \tan^{-1} 1 \) at the 400th term !!!

\[
\pi \approx 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots - \frac{1}{799}\right). \text{ The convergence of } \tan^{-1} 1 \text{ is too slow for practical value.}
\]

Example: Estimate \( e \) to 4 decimal place accuracy.

Consider the Taylor series of \( e^x \) about \( x = 0 \).

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ on } (-\infty, \infty). \text{ At } x = 1
\]

\[
e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} \approx 2.7183
\]
\[ e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \quad R_n(1) = \frac{f^{(n+1)}(c)1^{n+1}}{(n+1)!} = \frac{e^c}{(n+1)!}, \text{ c between 0 and 1}. \]

\[ R_n(1) < \frac{e}{(n+1)!} < \frac{3}{(n+1)!} < 5 \times 10^{-5} \text{ for 4 decimal place accuracy} \]

\[ (n+1)! > 60,000 \]

\[ n \geq 8 . \text{ Truncate the Taylor series for } e^1 \text{ at degree 8 term for 4 decimal place accuracy.} \]

\[ e \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{8!} = 2.71828 . \text{ The estimate is actually accurate to 5 decimal places.} \]

Example: Your homework is to redo the estimate for \( \sin 5 \) by noting that \( 5 \) radians is close to \( 3\pi/2 \).

Use the Taylor series of \( \sin x \) about \( 3\pi/2 \). Note that at \( x = 5 \) the Taylor series of \( \sin x \) about \( 3\pi/2 \) is an alternating series because \( x - 3\pi/2 > 0 \). It also meets the conditions of the Alternating Series Test. The magnitude of the truncation error is bounded by the magnitude of the next term in the Taylor series. Calculate how many terms you need for the same 12 place accuracy as before. \( |S - S_n| < 5 \times 10^{-13} \). You should get truncate at the 6th term which in this case is the \( (x - 3\pi/2)^{10} \) term. The 10th order Taylor polynomial is the estimation of \( \sin 5 \) for 12 decimal place accuracy. For homework problems see

http://voyager.dvc.edu/~lmonth/Calc2/Hw5Series.pdf