Mathematics is often referred to as a language with its own vocabulary and rules of grammar; one of the basic building blocks of the language of mathematics is the language of sets. Becoming familiar with the terms and symbols and learning to use them correctly will help you throughout your study of mathematics.

When attempting to make precise definitions, we have to begin by assuming that some terms are already understood. Many of the terms will be in your English vocabulary and only need a new mathematical interpretation. The following paragraph may give you some idea about the meanings of several of the terms used in set theory. They will be explained again later.

You are a member of the set of students at your College. The readers of this handout form a subset of the entire set of students or the universal set. The set of those students who do not read this handout form the complement of the set who do. If each person reading this has his or her own copy, then there is a one-to-one correspondence between the set of readers and the set of handouts being read.

We start this discussion by assuming that you know what is meant by a set (a collection or group) of elements and also what is meant by a particular element being a member of (belonging to) a set. The preceding paragraph may have given you some idea of what the terms subset, universal set, complement, and one-to-one correspondence mean, and we will give precise definitions after introducing some set notation.

We will usually describe a set by listing or describing its elements inside brackets. Here are some examples:

<table>
<thead>
<tr>
<th>&quot;Word Description&quot;</th>
<th>&quot;Set Notation&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) the set of whole numbers less than 5</td>
<td>{0,1,2,3,4}</td>
</tr>
<tr>
<td>b) the set of whole numbers</td>
<td>{0,1,2,3,4,...}</td>
</tr>
<tr>
<td>c) the set of letters in the word rate</td>
<td>{r,a,t,e} = {a,e,r,t}</td>
</tr>
<tr>
<td>d) the set of letters in the word tear</td>
<td>{t,e,a,r} = {a,e,r,t}</td>
</tr>
</tbody>
</table>

The elements of a set are the members. The membership symbol is \(\in\). We write 2 \(\in\) \{0,1,2,3,4\} and say, "2 is an element of the set," or "2 is a member of the set," or "2 is in the set." In general, the order in which the elements of a set are listed does not matter. However, when three dots are used to indicate that the list continues according to some pattern, then the order becomes important so that the pattern can be seen and used to identify the missing elements. The sets in examples (c) and (d) have exactly the same elements; they are equal.

It is often convenient to give sets names; capital letters are usually used. Consider these sets of numbers.

\[
\begin{align*}
W &= \{0,1,2,3,4,\ldots\} \\
E &= \{0,2,4,6,8,\ldots\} \\
F &= \{1,3,5,\ldots,15\} \\
G &= \{0,1,2,3,4\}
\end{align*}
\]

The sets W and E have no last element; they continue indefinitely and are called infinite sets. There is no rule about how many numbers must be included before the three dots are written, but there should be enough for the pattern to be recognized. Set F has just a few elements that are not listed. (Set F could also have been written F = \{1,3,5,7,9,11,13,15\}. Sets G and F are finite sets since we can actually count the number of elements in each.
Another way to write sets is to use set-builder notation. In this case set $E$ would be written

$$E = \{x | x \text{ is an even whole number}\}.$$ 

This is read:

$$E = \{x \mid x \text{ is an even whole number}\}$$

"$E$ is the set of all $x$, such that $x$ is an even whole number."

Note that the vertical bar is read, "such that."

Equivalently,

$$E = \{x \in W | x \text{ is even}\}$$

This can be read:

$$E = \{x \in W \mid x \text{ is even}\}$$

"$E$ is the set of all whole numbers $x$ such that $x$ is even,"

or more simply, "the set of all even whole numbers." (W is the symbol or name commonly used for the set of whole numbers.)

To indicate that an element is not a member of the set, we put a slash through the "element of" symbol $\notin$. For example, $1 \notin \{2,3,4,5,6,7\}$ and $5 \notin \{0,10,20\}$.

There are two special sets that need to be considered. The first is the set that contains every element under consideration. It is called the universal set and is often denoted by $U$. The other is the set with no elements; it is called the empty set or null set and is denoted by $\{\}$ or $\emptyset$.

One of the most important relationships between sets is the notion of a subset of a set. We will say that a set $A$ is a subset $B$, denoted by $A \subseteq B$, if every element of $A$ is also an element of $B$. The set $A$ is said to be a proper subset of $B$ if $A$ is a subset of $B$ but $A$ is not equal to $B$.

Consider the universal set $U = \{2,4,6,8,10\}$. Then the sets to be considered will have members from this set; they will be subsets of the universal set. Here are some examples of sets in this universe:

- $A = \{2,4,6\}$
- $B = \{2,4,6,8\}$
- $C = \{2,6,10\}$

Note that $A \subseteq U$, $B \subseteq U$, and $C \subseteq U$ since all the elements of $A$, $B$, and $C$ are contained in $U$. Also $A \subseteq B$, since every element in $A$ is also in $B$. However, $A$ is not a subset of $C$ since $4 \in A$ but $4 \notin C$. Likewise $C \not\subseteq B$ since $10 \in C$ but $10 \notin B$.

Do not confuse the notions of "element" and "subset." That is, $2 \in C$, but the number $2$ is not a subset of $C$ ($2 \not\in C$). On the other hand, the set whose only member is $2$, written $\{2\}$, is a subset of $C$ ($\{2\} \subseteq C$). Just how many subsets does $C$ have? Since $C = \{2,6,10\}$, we can list them:

$$\{\}, \{2\}, \{6\}, \{10\}, \{2,6\}, \{2,10\}, \{6,10\}, \{2,6,10\}$$

The number of elements in a set is called the cardinality of the set. The cardinality of the empty set is 0 since it has no elements. If a set is denoted by $A$, then its cardinality will be denoted by $n(A)$. For example, for $C = \{2,6,10\}$, $n(C) = 3$. Two different sets can have the same cardinality; in this case we say there is a one to one correspondence between the two sets. That is, we can "match up" the elements of one set with the elements of the other, one by one.

"A picture is worth a thousand words," and one useful way to "picture" sets and their relationships is with figures called Venn diagrams. In a Venn diagram the universal set is frequently represented by a rectangle with the proper sets in the universe represented by circles.
We can illustrate relationships between sets with appropriate Venn diagrams.

When A and B have no elements in common, we say they are disjoint. If the sets have elements in common, then they overlap or intersect.

Often we are given two sets X and Y, and we don't know if or how they are related. In this case we draw a general figure.
Let's label the different regions of this diagram:

(FIGURE 5)

We have to be careful when we make a general diagram to remember that there are several possibilities for each region. For example, just because we have drawn X and Y as overlapping sets does not mean that they must overlap. In fact, we have the following possibilities:
- X or Y or both may be the empty set.
- If \( X \subseteq Y \), then region 2 is empty.
- If \( Y \subseteq X \), then region 4 is empty.
- If \( X = Y \), then both regions 2 and 4 are empty.
- If X and Y are disjoint, then region 3 is empty.

Three common operations are performed on sets: union, intersection, and complementation.

Union is the operation that combines two sets; the union of the sets A and B is the set that consists of all the elements of A and in addition all the elements of B. The symbol for union is \( \cup \) and we write \( A \cup B \).

The intersection of the sets A and B is the set containing all the elements that are common to both sets and is denoted \( A \cap B \).

The complement of a set A is the set of all elements in U (the universe) and not in A. We will denote the complement of a set A by adding a prime: \( A' \). (Other books will use may use other notations, such as \( \sim A \) or \( A' \).)

Using set notation we have these definitions:
- The union of the sets A and B = \( A \cup B = \{x \mid x \in A \text{ or } x \in B\} \)
- The intersection of A and B = \( A \cap B = \{x \mid x \in A \text{ and } x \in B\} \)
- The complement of a set A = \( A' = \{x \mid x \in U \text{ and } x \notin A\} \)

Let's recall the sets we considered on page 2.
- \( U = \{2,4,6,8,10\} \)
- \( A = \{2,4,6\} \)
- \( B = \{2,4,6,8\} \)
- \( C = \{2,6,10\} \)

For the given sets A, B, and C and universe U, we have:
- \( A \cup B = \{2,4,6,8\} \)
- \( A \cap B = \{2,4,6\} \)
- \( A' = \{8,10\} \)
- \( A \cup C = \{2,4,6,10\} \)
- \( A \cap C = \{2,6\} \)
- \( B' = \{10\} \)
- \( B \cup C = \{2,4,6,8,10\} \)
- \( B \cap C = \{2,6\} \)
- \( C' = \{4,8\} \)
We can also draw Venn diagrams to describe each of these operations. When we illustrate the union of two sets A and B, first shade A and then shade B. The union is all the parts that have been shaded.

*(FIGURE 6)* Venn diagram showing the union of the sets A and B

\[ \text{A} \cup \text{B} \]

Intersection also can be illustrated using the Venn diagram. This time shade the set A in one direction or color and then shade the set B in another direction or color. The set \( A \cap B \) is the set that has been shaded twice.

*(FIGURE 7)* Venn diagram showing the intersection of A and B.

\[ A \cap B \]

Complementation, too, can be shown using a Venn diagram. In this case, the area outside the given set is shaded.

*(FIGURE 8)* Venn diagram showing the complement of the set A.

\[ A' \cup U \]
PROBLEM SET

List all subsets of each set given in Problems 1-5
1. \( \emptyset \)  
2. \{1,2\}  
3. \{2,4\}  
4. \{2,4,6\}  
5. \{2,5,7\}

How many subsets do each of these sets have?
6. \{1,2,3,4\}  
7. \{a,e,i,o,u\}  
8. \{1,3,5,7,9,11\}

Use the following sets to answer the questions that follow.
\[ U = \{1,2,3,4,5,6,7,8,9,10\} \]
\[ X = \{2,4,6,8,10\} \]
\[ Y = \{5,6,7\} \]
\[ Z = \{1,3,5,7,9\} \]

TRUE OR FALSE?
9. \( X \subseteq Y \)  
10. \( X \subseteq Z \)  
13. \( X \text{ and } Z \) are disjoint.  
14. \( 2 \in X \)  
15. \( \{1,3\} \in Z \)  
12. \( 2 \in Y \)  
16. \( \{\} \subseteq X \)

List all the members of each set in problems 17 - 25.
17. \( X \cap Z \)  
18. \( X \cup Y \)  
19. \( X' \)  
20. \( Y' \)  
21. \( X \cup Z \)  
22. \( X \cap Y \)  
23. \( Y \cap Z \)  
24. \( Y \cup Z \)  
25. \( Z' \)

Draw a Venn diagram for each relationship in problems 26 – 31.
26. \( X \cap Z \)  
27. \( X \cup Y \)  
28. \( Y \cup Z \)  
29. \( X \cup Z \)  
30. \( X \cap Y \)  
31. \( Y \cap Z \)  
32. \( X' \)  
33. \( Y' \)  
34. \( Z' \)

List the members of each of these sets.
35. \( \{x \mid x \in W \text{ and } x < 6\} \)
36. \( \{x \mid x \in W \text{ and } -1 < x < 4\} \)
37. \( \{x \mid x \in W \text{ and } x \text{ is a multiple of 3}\} \cap \{x \mid x \in W \text{ and } x < 10\} \)
38. \( \{x \mid x \in W \text{ and } -1 < x < 4\} \cup \{x \mid x \in W \text{ and } -5 < x < 2\} \)
Answers to the odd problems

1. $\emptyset$
3. $\emptyset$, {2}, {4}, {2,4}
5. $\emptyset$, {2}, {5}, {7}, {2,7}, {2,5}, {5,7}, {2,5,7}
7. 32
9. false
11. false
13. true
15. false
17. none
19. {1,3,5,7,9}
21. {1,2,3,4,5,6,7,8,9,10}
23. {5,7}
25. {2,4,6,8,10}

27. [Diagram showing sets $X$, $Y$, and $U$ with elements 2, 4, 6, 8, 10, and 5, 7]
29. [Diagram showing sets $X$ and $Z$ with elements 2, 4, 6, 8, 10 and 1, 3, 5, 7, 9]
31. [Diagram showing sets $Y$, $Z$, and $U$ with elements 6, 5, 7, and 1, 3, 9]
33. [Diagram showing sets $Y$ and $U$ with elements 5, 6, 7]

35. {0,1,2,3,4,5}
37. {0,3,6,9}